Simultaneous equations

A LINEAR SIMULTANEOUS EQUATIONS

Linear simultaneous equations are two equations containing two unknowns.

When we solve these problems we are trying to find the solution which is common to both equations.

Notice that in $\begin{cases} x+y=9 \\ 2x+3y=21 \end{cases}$, if $x=6$ and $y=3$ then:

- $x+y=(6)+(3)=9$ ✓ i.e., the equation is satisfied
- $2x+3y=2(6)+3(3)=12+9=21$ ✓ i.e., the equation is satisfied.

So, $x=6$ and $y=3$ is the **solution** to the simultaneous equations $\begin{cases} x+y=9 \\ 2x+3y=21 \end{cases}$.

The solutions to **linear simultaneous equations** can be found by **trial and error** (a little tedious) or **graphically** as in **Chapter 7** (which can be inaccurate if solutions are not integers).

However, because of the limitations of these methods, other algebraic methods are used.

SOLUTION BY SUBSTITUTION

The method of **solution by substitution** is used when at least one equation is given with either x or y as the **subject** of the formula.

Example 1 Self Tutor

Solve simultaneously, by substitution: $y=9-x$
$2x+3y=21$

Notice that $9-x$ is substituted for y in the other equation.

$$y=9-x \quad \text{...... (1)}$$
$$2x+3y=21 \quad \text{...... (2)}$$

Since $y=9-x$, then $2x+3(9-x)=21$

$\therefore\ 2x+27-3x=21$

$\therefore\ 27-x=21$

$\therefore\ x=6$

and so, when $x=6$, $y=9-6$ {substituting $x=6$ into (1)}

$\therefore\ y=3$.

Solution is: $x=6$, $y=3$.

Check: (1) $3=9-6$ ✓ (2) $2(6)+3(3)=12+9=21$ ✓

Example 2 — Self Tutor

Solve simultaneously, by substitution: $2y - x = 2$
$x = 1 + 8y$

$$2y - x = 2 \quad ...... (1)$$
$$x = 1 + 8y \quad ...... (2)$$

Substituting (2) into (1) gives

$$\therefore \quad 2y - (1 + 8y) = 2$$
$$\therefore \quad 2y - 1 - 8y = 2$$
$$\therefore \quad -6y - 1 = 2$$
$$\therefore \quad -6y = 3$$
$$\therefore \quad y = -\tfrac{1}{2}$$

Substituting $y = -\frac{1}{2}$ into (2) gives

$$x = 1 + 8 \times -\tfrac{1}{2} = -3.$$

The solution is $x = -3$, $y = -\frac{1}{2}$.

Check: (1) $2(-\frac{1}{2}) - (-3) = -1 + 3 = 2$ ✓
(2) $1 + 8(-\frac{1}{2}) = 1 - 4 = -3$ ✓

EXERCISE 17A.1

1 Solve simultaneously, using substitution:

a $x = 8 - 2y$, $2x + 3y = 13$
b $y = 4 + x$, $5x - 3y = 0$
c $x = -10 - 2y$, $3y - 2x = -22$
d $x = -1 + 2y$, $x = 9 - 2y$
e $3x - 2y = 8$, $x = 3y + 12$
f $x + 2y = 8$, $y = 7 - 2x$

2 Use the substitution method to solve simultaneously:

a $x = -1 - 2y$, $2x - 3y = 12$
b $y = 3 - 2x$, $y = 3x + 1$
c $x = 3y - 9$, $5x + 2y = 23$
d $y = 5x$, $7x - 2y = 3$
e $x = -2 - 3y$, $3x - 2y = -17$
f $3x - 5y = 26$, $y = 4x - 12$

3 **a** Try to solve by substitution, $y = 3x + 1$ and $y = 3x + 4$.
b What is the simultaneous solution for the equations in **a**?

4 **a** Try to solve by substitution, $y = 3x + 1$ and $2y = 6x + 2$.
b How many simultaneous solutions do the equations in **a** have?

SOLUTION BY ELIMINATION

In many problems which require the simultaneous solution of linear equations, each equation will be of the form $ax + by = c$. Solution by substitution is often tedious in such situations and the method of elimination of one of the variables is preferred.

In the method of **elimination**, we eliminate (remove) one of the variables by making the coefficients of x (or y) the **same size** but **opposite in sign** and then **adding** the equations. This has the effect of **eliminating** one of the variables.

Example 3 **Self Tutor**

Solve simultaneously, by elimination: $4x + 3y = 2$ ...... (1)
$x - 3y = 8$ ...... (2)

We **sum** the LHS's and the RHS's to get an equation which contains x only.

$$\begin{array}{rl} & 4x + 3y = 2 \\ + & \;\; x - 3y = 8 \\ \hline & 5x \qquad = 10 \quad \{\text{on adding the equations}\} \\ \therefore & \;\; x = 2 \quad \{\text{dividing both sides by } 5\} \end{array}$$

Let $x = 2$ in (1) $\therefore\ 4(2) + 3y = 2$

$\therefore\ 8 + 3y = 2$

$\therefore\ 3y = 2 - 8$ {subtracting 8 from both sides}

$\therefore\ 3y = -6$

$\therefore\ y = -2$ {dividing by 3 on both sides}

i.e., $x = 2$ and $y = -2$

Check: in (2): $(2) - 3(-2) = 2 + 6 = 8$ ✓

The method of elimination uses the fact that: If $a = b$ and $c = d$ then $a + c = b + d$.

EXERCISE 17A.2

1 What equation results when the following are added vertically?

a $5x + 3y = 12$
$x - 3y = -6$

b $2x + 5y = -4$
$-2x - 6y = 12$

c $4x - 6y = 9$
$x + 6y = -2$

d $12x + 15y = 33$
$-18x - 15y = -63$

e $5x + 6y = 12$
$-5x + 2y = -8$

f $-7x + y = -5$
$7x - 3y = -11$

2 Solve the following using the method of elimination:

a $2x + y = 3$
$3x - y = 7$

b $4x + 3y = 7$
$6x - 3y = -27$

c $2x + 5y = 16$
$-2x - 7y = -20$

d $3x + 5y = -11$
$-3x - 2y = 8$

e $4x - 7y = 41$
$3x + 7y = -6$

f $-4x + 3y = -25$
$4x - 5y = 31$

In problems where the coefficients of x (or y) are **not** the **same size** or **opposite in sign**, we may have to **multiply** each equation by a number to enable us to **eliminate** one variable.

Example 4 **Self Tutor**

Solve simultaneously, by elimination: $3x + 2y = 7$
$2x - 5y = 11$

$$3x + 2y = 7 \quad \text{......(1)}$$
$$2x - 5y = 11 \quad \text{......(2)}$$

We can eliminate y by multiplying (1) by 5 and (2) by 2.

$$\begin{aligned} \therefore \quad 15x + 10y &= 35 \\ + \quad 4x - 10y &= 22 \\ \hline \therefore \quad 19x \qquad &= 57 \qquad \text{\{on adding the equations\}} \\ \therefore \quad x &= 3 \qquad \text{\{dividing both sides by 19\}} \end{aligned}$$

Substituting $x = 3$ into equation (1) gives

$$\begin{aligned} 3(3) + 2y &= 7 \\ \therefore \quad 9 + 2y &= 7 \\ \therefore \quad 2y &= -2 \\ \therefore \quad y &= -1 \end{aligned}$$

So, the solution is: $x = 3$, $y = -1$. *Check:* $3(3) + 2(-1) = 9 - 2 = 7$ ✓
$2(3) - 5(-1) = 6 + 5 = 11$ ✓

3 Give the equation that results when both sides of the equation:

a $3x + 4y = 2$ are multiplied by 3
b $x - 4y = 7$ are multiplied by -2
c $5x - y = -3$ are multiplied by 5
d $7x + 3y = -4$ are multiplied by -3
e $-2x - 5y = 1$ are multiplied by -4
f $3x - y = -1$ are multiplied by -1

Example 5 **Self Tutor**

Solve by elimination: $3x + 4y = 14$
$4x + 5y = 17$

$$3x + 4y = 14 \text{ ...... (1)}$$
$$4x + 5y = 17 \text{ ...... (2)}$$

To eliminate x, multiply both sides of

$$\begin{aligned} \text{(1) by 4:} \quad 12x + 16y &= 56 \quad \text{...... (3)} \\ \text{(2) by } -3\text{:} \quad -12x - 15y &= -51 \quad \text{...... (4)} \\ \hline y &= 5 \quad \text{\{on adding (3) and (4)\}} \end{aligned}$$

and substituting $y = 5$ into (2) gives

$$\begin{aligned} 4x + 5(5) &= 17 \\ \therefore \quad 4x + 25 &= 17 \\ \therefore \quad 4x &= -8 \\ \therefore \quad x &= -2 \end{aligned}$$

Check:
(1) $3(-2) + 4(5) = (-6) + 20 = 14$ ✓
(2) $4(-2) + 5(5) = (-8) + 25 = 17$ ✓

Thus $x = -2$ and $y = 5$.

WHAT TO ELIMINATE

There is always a choice whether to eliminate x or y, so our choice depends on which variable is easier to eliminate.

In **Example 5**, try to solve by multiplying (1) by 5 and (2) by -4. This eliminates y rather than x. The final solutions should be the same.

4 Solve the following using the method of elimination:

a $4x-3y=6$ $-2x+5y=4$	**b** $2x-y=9$ $x+4y=36$	**c** $3x+4y=6$ $x-3y=-11$
d $2x+3y=7$ $3x-2y=4$	**e** $4x-3y=6$ $6x+7y=32$	**f** $7x-3y=29$ $3x+4y+14=0$
g $2x+5y=20$ $3x+2y=19$	**h** $3x-2y=10$ $4x+3y=19$	**i** $3x+4y+11=0$ $5x+6y+7=0$

5 Use the method of elimination to attempt to solve:

a $3x+y=8$
$6x+2y=16$

b $2x+5y=8$
$4x+10y=-1$

Comment?

B PROBLEM SOLVING

Many problems can be described mathematically by a **pair of linear equations**, i.e., two equations of the form $ax+by=c$, where x and y are the two variables (unknowns).

We have already seen an example of this in the **Investigation** on page **350**.

Once the equations are formed, they can then be solved simultaneously and the original problem can be solved. The following method is recommended:

Step 1: Decide on the two unknowns; call them x and y, say. Do not forget the units.
Step 2: Write down **two** equations connecting x and y.
Step 3: Solve the equations simultaneously.
Step 4: Check your solutions with the original data given.
Step 5: Give your answer in sentence form.

(**Note:** The form of the original equations will help you decide whether to use the substitution method, or the elimination method.)

Example 6 **Self Tutor**

Two numbers have a sum of 45 and a difference of 13. Find the numbers.

Let x and y be the unknown numbers, where $x > y$.

Then $x+y=45$ ......(1) {'sum' means add}
and $x-y=13$ ......(2) {'difference' means subtract}

$2x = 58$ {adding (1) and (2)}
$\therefore\ x=29$ {dividing both sides by 2}

and substituting into (1),
$29+y=45$
$\therefore\ y=16$

Check:
(1) $29+16=45$ ✓
(2) $29-16=13$ ✓

The numbers are 29 and 16.

When solving problems with simultaneous equations we must find two equations containing two unknowns.

EXERCISE 17B

1 The sum of two numbers is 47 and their difference is 14. Find the numbers.

2 Find two numbers with sum 28 and half their difference 2.

3 The larger of two numbers is four times the smaller and their sum is 85. Find the two numbers.

Example 7 **Self Tutor**

When shopping in the West Indies, 5 oranges and 14 bananas cost me \$1.30, and 8 oranges and 9 bananas cost \$1.41. Find the cost of each orange and each banana.

Let each orange cost x cents and each banana cost y cents.

$$\therefore \quad 5x + 14y = 130 \text{ ......(1)}$$
$$8x + 9y = 141 \text{ ......(2)}$$

{**Note:** Units must be the same on both sides of each equation i.e., cents}

To eliminate x, we multiply (1) by 8 and (2) by -5.

$$\therefore \quad 40x + 112y = 1040 \text{ .....(3)}$$
$$\underline{-40x - 45y = -705} \text{ .....(4)}$$

adding (3) and (4) $\quad 67y = 335$

$\therefore \quad y = 5 \qquad$ {dividing both sides by 67}

and substituting in (2)

$$8x + 9 \times 5 = 141$$
$$\therefore \quad 8x = 141 - 45$$
$$\therefore \quad 8x = 96$$
$$\therefore \quad x = 12 \qquad \text{\{dividing both sides by 8\}}$$

Check: $\quad 5 \times 12 + 14 \times 5 = 60 + 70 = 130$ ✓

$8 \times 12 + 9 \times 5 = 96 + 45 = 141$ ✓

Thus oranges cost 12 cents, bananas cost 5 cents each.

4 Five pencils and 6 biros cost a total of \$4.64, whereas 7 pencils and 3 biros cost a total of \$3.58. Find the cost of each item.

5 Seven toffees and three chocolates cost a total of \$1.68, whereas four toffees and five chocolates cost a total of \$1.65. Find the cost of each of the sweets.

Example 8 **Self Tutor**

In my pocket I have only 5-cent and 10-cent coins. How many of each type of coin do I have if I have 24 coins altogether and their total value is \$1.55?

Let x be the number of 5-cent coins and y be the number of 10-cent coins.

$\therefore \quad x + y = 24 \quad$ ......(1) $\quad$ {the total number of coins}

and $\quad 5x + 10y = 155 \quad$ ......(2) $\quad$ {the total value of coins}

Multiplying (1) by -5 gives

$$-5x - 5y = -120 \text{ ......(3)}$$
$$\underline{5x + 10y = 155} \text{ ......(2)}$$
$$\therefore \quad 5y = 35 \qquad \text{\{adding (3) and (2)\}}$$

$\therefore \quad y = 7 \qquad$ {dividing both sides by 5}

and substituting in (1) gives

$x + 7 = 24$

$\therefore \quad x = 17$

Check: $17 + 7 = 24$ ✓

$5 \times 17 + 10 \times 7 = 85 + 70 = 155.$ ✓

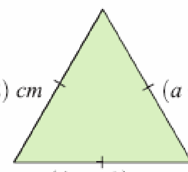
Thus I have 17 five cent coins and 7 ten cent coins.

6 I collect only 50-cent and \$1 coins. My collection consists of 43 coins and their total value is \$35. How many of each coin type do I have?

7 Amy and Michelle have \$29.40 between them and Amy's money is three quarters of Michelle's. How much money does each have?

8 Margarine is sold in either 250 g or 400 g packs. A supermarket manager ordered 19.6 kg of margarine and received 58 packs. How many of each type did the manager receive?

9 Given that the triangle alongside is equilateral, find a and b.



10 A rectangle has perimeter 32 cm. If 3 cm is taken from the length and added to the width, the rectangle becomes a square. Find the dimensions of the original rectangle.

HARDER PROBLEMS (EXTENSION)

Example 9 **Self Tutor**

A boat travels 24 km upstream in 4 hours. The return trip downstream takes only 3 hours. Given that the speed of the current is constant throughout the entire trip, what was:

a the speed of the current **b** the speed of the boat in still water?

Let the speed of the boat in still water be x km per hour, and the speed of the current be y km per hour.

$$\text{speed upstream} = \frac{24 \text{ km}}{4 \text{ h}} = 6 \text{ km/h}$$

$$\text{speed downstream} = \frac{24 \text{ km}}{3 \text{ h}} = 8 \text{ km/h}$$

boat speed − current speed = actual speed (upstream)

$\therefore \quad x - y = 6 \qquad$ ..... (1)

and boat speed + current speed = actual speed (downstream)

$\therefore \quad x + y = 8 \quad ..... (2)$

adding (1) and (2): $2x = 14$

$\therefore \quad x = 7$

and substituting in (1): $7 - y = 6$

$\therefore \quad y = 1$

Check: $7 - 1 = 6$ ✓ and $7 + 1 = 8$ ✓

a Current speed is 1 km/h. **b** Boat speed in still water is 7 km/h.

11 A motor boat travels 12 km/h upstream against the current and 18 km/h downstream with the current. Find the speed of the current and the speed of the motor boat in still water.

12 A jet plane made a 4000 km trip with the wind, in 4 hours, but required 5 hours to make the return trip. Given that the speed of the wind was constant throughout the entire trip, what was the speed of the wind and what was the average speed of the plane in still air?

13 A man on foot covers the 25 km between two towns in $3\frac{3}{4}$ hours. He walks at 4 km/h for the first part of the journey and runs at 12 km/h for the remaining part.

a How far did he run? **b** For how long was he running?

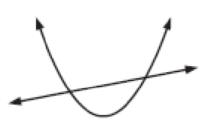
14 Explain why any two digit number can be written in the form $10a + b$. Hence, solve the following problem:

A number consists of two digits which add up to 9. When the digits are reversed, the original number is decreased by 45. What was the original number?

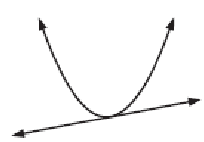
C WHERE FUNCTIONS MEET

Consider the graphs of a quadratic function and a linear function on the same set of axes.

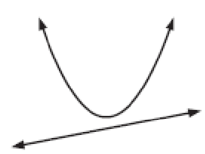
Notice that we could have:



cutting
(2 points of intersection)



touching
(1 point of intersection)



missing
(no points of intersection)

When the graphs meet, the coordinates of their points of intersection can be found by *solving the two equations simultaneously.*

Example 10 — Self Tutor

Find algebraically the coordinates of the points of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$y = x^2 - x - 18$ meets $y = x - 3$ where

$$x^2 - x - 18 = x - 3$$
$$\therefore \quad x^2 - 2x - 15 = 0 \qquad \{\text{RHS} = 0\}$$
$$\therefore \quad (x-5)(x+3) = 0 \qquad \{\text{factorising}\}$$
$$\therefore \quad x = 5 \text{ or } -3$$

Substituting into $y = x - 3$, when $x = 5$, $y = 2$ and when $x = -3$, $y = -6$.

$\therefore$ graphs meet at (5, 2) and (−3, −6).

EXERCISE 17C

1 Find algebraically, the coordinates of the point(s) of intersection of the graphs with equations:

- **a** $y = x^2 - 2x + 8$ and $y = x + 6$
- **b** $y = -x^2 + 3x + 9$ and $y = 2x - 3$
- **c** $y = x^2 - 4x + 3$ and $y = 2x - 6$
- **d** $y = x^2 - 3x + 2$ and $y = x - 3$
- **e** $y = -x^2 + 4x - 7$ and $y = 5x - 4$
- **f** $y = x^2 - 5x + 9$ and $y = 3x - 7$

2 Use a **graphing package** or a **graphics calculator** to find the coordinates of the points of intersection (to two decimal places) of the graphs with equations:

- **a** $y = x^2 - 3x + 7$ and $y = x + 5$
- **b** $y = x^2 - 5x + 2$ and $y = x - 7$
- **c** $y = -x^2 - 2x + 4$ and $y = x + 8$
- **d** $y = -x^2 + 4x - 2$ and $y = 5x - 6$

3 Use a **graphing package** or a **graphics calculator** to find the coordinates of the points of intersection (to two decimal places) of the graphs with equations:

- **a** $y = 2x + 7$ and $y = \dfrac{3}{x}$
- **b** $y = \dfrac{4}{x^2}$ and $y = x + 5$
- **c** $y = x^2 + 5x - 1$ and $y = x - 5$
- **d** $y = x^2 - 3x + 1$ and $y = 3 - x^2$
- **e** $y = \dfrac{3}{x-1}$ and $y = x^2 - 2$
- **f** $y = \dfrac{3}{x(x-2)}$ and $y = x^2 + 2x - 8$

REVIEW SET 17A

1 Solve by using the method of substitution: $y = 2x - 7$ and $2x + 3y = 11$.

2 Solve the following simultaneous equations:
a $y = 2x - 5$, $3x - 2y = 11$
b $3x + 5y = 1$, $4x - 3y = 11$

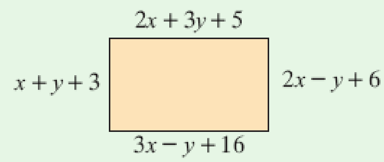
3 Flour is sold in 5 kg and 2 kg packets. The 5 kg packets cost \$2.75 and the 2 kg packets cost \$1.25 each. If I bought 67 kg of flour and the total cost was \$38.50, how many of each kind of packet did I buy?

4 Find the coordinates of any point where these line pairs meet:

a $y = 3x + 2$
$y = 3x - 1$

b $2x - 3y = 18$
$4x + 5y = -8$

5 For the given rectangle, find its perimeter. Your answer must not contain x and y.



6 Sally has only 10-cent and 50-cent coins in her purse. She has 21 coins altogether with a total value of \$5.30. How many of each coin type does she have?

7 Find algebraically where the line with equation $y = 2x - 3$ meets the parabola with equation $y = 2x^2 - 3x - 10$.

8 $33x + 67y = 533$
$67x + 33y = 567$

Solve algebraically for x and y.
(**Hint:** Look carefully at the equations.)

REVIEW SET 17B

1 Solve using the method of substitution:

$y = 11 - 3x$
$4x + 3y = -7$

2 Solve the following simultaneous equations:

a $3x - 2y = 16$
$y = 2x - 10$

b $3x - 5y = 11$
$4x + 3y = 5$

3 A bus company uses two different sized buses. If the company uses 7 small buses and 5 large buses to transport 331 people, but needs 4 small buses and 9 large buses to carry 398 people, determine the number of people each bus can carry.

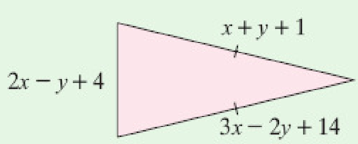
4 Find the coordinates of any point where these line pairs meet:

a $y = 2x + 5$
$y = 2x + 15$

b $3x + 7y = -6$
$6x + 5y = 15$

5 Orange juice can be purchased in 2 L cartons or in 600 mL bottles. The 2 L cartons cost \$1.50 each and the 600 mL bottles cost \$0.60 each. A consumer purchases 73 L of orange juice and his total cost was \$57. How many of each container did the consumer buy?

6



For this *isosceles* triangle the perimeter is 29 cm.

Find the length of the equal sides. Your answer must not contain x and y.

7 Find algebraically the points of intersection of the parabolas with equations $y = 2x^2 + 3x - 1$ and $y = 5 - x^2$.

8 Solve the system $\begin{cases} x + y + z = 5 \\ x - 2y + z = 2 \\ 2x + y - 2z = 1 \end{cases}$ simultaneously.