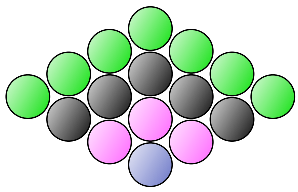
Sequences

1. Number sequences

Consider the illustrated pattern of circles:

The first layer has just one blue ball.
The second layer has three pink balls.
The third layer has five black balls.
The fourth layer has seven green balls.



If we let u_n represent the number of balls in the nth layer, then $u_1 = 1$, $u_2 = 3$, $u_3 = 5$, and $u_4 = 7$.

The pattern could be continued forever, generating the **sequence** of numbers:

1, 3, 5, 7, 9, 11, 13, 15, 17, ...

The string of dots indicates that the pattern continues forever.

A **sequence** is a set of terms, in a definite order, where the terms are obtained by some rule.

The sequence for the pattern of balls can be specified:

- **using words**: "The set of all odd numbers starting with 1".
- **using an explicit formula**: $u_n = 2n - 1$ generates all terms. u_n is called the nth *term* or the *general term*.
- **using a recursive formula**: $u_n = 1$ and $u_{n+1} = u_n + 2$ for all $n \geq 1$.

  Check: $u_1 = 1$

  $u_2 = u_1 + 2 = 1 + 2 = 3 \quad \checkmark$

  $u_3 = u_2 + 2 = 3 + 2 = 5 \quad \checkmark$

2. Arithmetic sequences

An **arithmetic sequence** is a sequence in which each term differs from the previous one by the same fixed number. We call this number the **common difference.**

For example: 1, 5, 9, 13, 17, . . . is arithmetic as $5 - 1 = 9 - 5 = 13 - 9$, etc. The difference is 4.

Likewise, 42, 37, 32, 27, . . . is arithmetic as $37 - 42 = 32 - 37 = 27 - 32$, etc. The difference is -5.

In the sequence 1, 5, 9, 13, 17, . . . notice that

$u_1 = 1$

$u_2 = 1 + 4 \qquad = 1 + 1 \cdot 4$

$u_3 = 1 + 4 + 4 + \qquad = 1 + 2 \cdot 4$

$u_4 = 1 + 4 + 4 + 4 \qquad = 1 + 3 \cdot 4,$ etc.

This suggests that:

If u_n is arithmetic then the nth term is $u_n = u_1 + (n-1)d$ where u_1 is the first term and d is the constant common difference.

Example 1

Consider the sequence 3, 9, 15, 21, 27, . . . :

a) Show that the sequence is arithmetic.

b) Find the formula for the general term u_n.

c) Find the 100th term of the sequence.

d) Is **i)** 489 **ii)** 1592 a member of the sequence?

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a) $9 - 3 = 6, \quad 15 - 9 = 6 \quad 21 - 15 = 6, \quad 27 - 21 = 6$

So, assuming that the pattern continues, consecutive terms differ by 6. Hence, the sequence is arithmetic with $u_1 = 3$ and $d = 6$.

b) $u_n = u_1 + (n-1)d \;\Rightarrow\; u_n = 3 + 6(n-1) \;\Rightarrow\; u_n = 6n - 3$

c) If $n = 100, \quad u_{100} = 6 \cdot 100 - 3 = 597$.

d) **i)** Let $u_n = 489 \;\Rightarrow\; 6n - 3 = 489 \;\Rightarrow\; n = 82$

489 is a term of the sequence. In fact it is the 82nd term.

ii) Let $u_n = 1592 \;\Rightarrow\; 6n - 3 = 1592 \;\Rightarrow\; n = 265\frac{5}{6}$

which is not possible as n is an integer. Hence 1592 cannot be a term.

Example 2

Find k given that $k+5$, -1 and $2k-1$ are consecutive terms of an arithmetic sequence, and hence find the terms.

Since the terms are consecutive,

$$\begin{aligned} -1-(k+5) &= (2k-1)-(-1) \qquad \text{equating common differences} \\ \therefore\quad -1-k-5 &= 2k-1+1 \\ \therefore\quad -k-6 &= 2k \\ \therefore\quad -6 &= 3k \\ \therefore\quad k &= -2 \end{aligned}$$

$\therefore$ the terms are $3, -1, -5$.

Example 3

Find the general term u_n for an arithmetic sequence given that $u_3 = 4$ and $u_7 = -24$.

$u_7 - u_3 = (u_1 + 6d) - (u_1 + 2d) = u_1 + 6d - u_1 - 2d = 4d$

But $\quad u_7 - u_3 = -24 - 4 = -28 \;\Rightarrow\; 4d = -28 \;\Rightarrow\; d = -7$

Now $\quad u_3 = u_1 + 2 \cdot (-7) \;\Rightarrow\; 4 = u_1 - 14 \;\Rightarrow\; u_1 = 18$

Hence $\quad u_n = 18 + (n-1)(-7) \;\Rightarrow\; u_n = 25 - 7n$

Exercises - Set A

1. Consider the sequence 4, 11, 18, 25, 32, . . .

   a) Show that the sequence is arithmetic. b) Find the formula for its general term.
   c) Find its 30th term. d) Is 340 a member?
   e) Is 738 a member?

2. Consider the sequence 67, 63, 59, 55, . . .

   a) Show that the sequence is arithmetic. b) Find the formula for its general term.
   c) Find its 60th term. d) Is -143 a member?
   e) Is 85 a member?

3. An arithmetic sequence is defined by $u_n = 11n - 7$.

   a) Find u_1 and d. b) Find the 37th term.
   c) What is the least term of the sequence which is greater than 250?

4. A sequence is defined by $u_n = \dfrac{21-4n}{2}$.

   a) Prove that the sequence is arithmetic. b) Find u_1 and d. c) Find u_{55}.
   d) For what values of n are the terms of the sequence less than -300?

5. Find k given the consecutive arithmetic terms:

   a) $31, \quad k, \quad 13$ b) $k, \quad 8, \quad k+11$ c) $k+2, \quad 2k+3, \quad 17$

6. Find the general term u_n for an arithmetic sequence given that:

   a) $u_4 = 37$ and $u_{10} = 67$.
   b) $u_5 = -10$ and $u_{12} = -38$.
   c) the fourth term is -4 and the fifteenth term is 29.
   d) the tenth and sixth terms are -16 and -13 respectively.

7. Consider the finite arithmetic sequence $3, 2\frac{1}{2}, 2\ldots, -6$.

   a) Find u_1 and d. b) How many terms does the sequence have?

8. An arithmetic sequence starts 17, 24, 31, 38, . . . What is the first term of the sequence to exceed 40 000?

3. Geometric sequences

A sequence is **geometric** if each term can be obtained from the previous one by multiplying by the same non-zero constant.

For example: 2, 6, 18, 54, . . . is a geometric sequence as $2 \cdot 3 = 6$ and $6 \cdot 3 = 18$ and $18 \cdot 3 = 54$.

Notice that

$u_2 = u_1 \cdot 3$

$u_3 = u_2 \cdot 3 = u_1 \cdot 3 \cdot 3 = u_1 \cdot 3^2$

$u_4 = u_3 \cdot 3 = u_1 \cdot 3^2 \cdot 3 = u_1 \cdot 3^3$

$u_5 = u_4 \cdot 3 = u_1 \cdot 3^3 \cdot 3 = u_1 \cdot 3^4$

This suggests the following algebraic definition:

If u_n is geometric then $u_n = u_1 \cdot r^{n-1}$ for all positive integers n.
u_1 is the first term and r is a constant called the **common ratio**.

Notice:

- r is called the common ratio because $\dfrac{u_{n+1}}{u_n} = r$ for all n.
- $2, 6, 18, 54, \ldots$ is geometric with $r = 3$.
- $2, -6, 18, -54, \ldots$ is geometric with $r = -3$.

Example 4

For the sequence 16, 8, 4, 2, 1 ...:
a) Show that the sequence is geometric.
b) Find the general term u_n.
c) Hence, find the 10th term as a fraction.

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a) $\dfrac{8}{16} = \dfrac{1}{2}, \quad \dfrac{4}{8} = \dfrac{1}{2}, \quad \dfrac{2}{4} = \dfrac{1}{2}, \quad \dfrac{1}{2} = \dfrac{1}{2}$

So, assuming the pattern continues, consecutive terms have a common ratio of $\dfrac{1}{2}$. Hence, the sequence is geometric with $u_1 = 16$ and $r = \dfrac{1}{2}$.

b) $u_n = u_1 \cdot r^{n-1} \;\Rightarrow\; u_n = 16 \cdot \left(\dfrac{1}{2}\right)^{n-1} = 2^4 \cdot (2^{-1})^{n-1} = 2^{4+(-n+1)} = 2^{5-n}$

c) $u_{10} = 2^{5-10} = 2^{-5} = \dfrac{1}{2^5} = \dfrac{1}{32}$

Example 5

$k-1, \quad k+2 \quad$ and $3k \quad$ are consecutive terms of a geometric sequence. Find k.

Equating common ratios gives $\quad \dfrac{3k}{k+2} = \dfrac{k+2}{k-1} \;\Rightarrow\; 3k(k-1) = (k+2)^2 \;\Rightarrow$

$3k^2 - 3k = k^2 + 4k + 4 \;\Rightarrow\; 2k^2 - 7k - 4 = 0 \;\Rightarrow\; k = 4$ or $k = -\dfrac{1}{2}$

Check: If $k = 4$, the terms are: $3, 6, 12 \qquad (r = 2)$

If $k = -\dfrac{1}{2}$, the terms are: $-\dfrac{3}{2}, \dfrac{3}{2}, -\dfrac{3}{2} \qquad (r = -1)$

Example 6

A geometric sequence has $u_2 = -5$ and $u_5 = 40$. Find its general term.

$u_2 = u_1 r = -5$ and $u_5 = u_1 r^4 = 40$, so $\dfrac{u_1 r^4}{u_1 r} = \dfrac{40}{-5} \Rightarrow r^3 = -8 \Rightarrow r = -2$

Then, $u_1(-2) = -5 \Rightarrow u_1 = \dfrac{5}{2}$, and $u_n = \dfrac{5}{2} \cdot (-2)^{n-1}$

Exercises - Set B

1. For the geometric sequence with first two terms given, find b and c:

   a) $3, 6, b, c, \ldots$ b) $8, 2, b, c, \ldots$ c) $15, -5, b, c, \ldots$

2. Consider the sequence 1, 3, 9, 27, ...

   a) Show that the sequence is geometric.

   b) Find u_n and hence find the 10th term.

3. Consider the sequence $40, -20, 10, -5 \ldots$

   a) Show that the sequence is geometric.

   b) Find u_n and hence find the 12th term as a fraction.

4. Show that the sequence $16, -4, 1, -0, 25, \ldots$ is geometric and hence find the 8th term as a decimal.

5. Find the general term of the geometric sequence: $3, 3\sqrt{2}, 6, 6\sqrt{2}, \ldots$

6. Find k given that the following are consecutive terms of a geometric sequence:

   a) $k, 2, 6$ b) $4, 6, k$ c) $k, 2\sqrt{2}, k^2$

   d) $3, k, 27$ e) $k, 3k, 10k + 7$ f) $k, k + 4, 8k + 2$

7. Find the general term u_n of the geometric sequence which has:

   a) $u_3 = 16$ and $u_8 = 512$ b) $u_3 = 32$ and $u_6 = -4$

   c) $u_7 = 24$ and $u_{15} = 384$ d) $u_3 = 3$ and $u_9 = \dfrac{3}{8}$

8. A geometric sequence has general term u_n with $u_3 = 12$ and $u_7 = \dfrac{3}{4}$. Find u_{12}.

9. u_n is the general term of a geometric sequence.

   a) If $u_2 = -2\frac{1}{2}$ and $u_5 = \dfrac{5}{16}$, find u_{10}.

   b) If $u_3 = 7$ and $u_8 = -7$, find u_{88}.

   c) If $u_3 = 18$ and $u_5 = 162$, find u_{11}.

4. Sum of n terms of an arithmetic sequence

Recall that if the first term of an arithmetic sequence is u_1 and the common difference is d, then the terms are: $u_1,\ u_1+d,\ u_1+2d,\ u_1+3d$, etc.

Suppose that u_n is the last term of the sequence. Then, the sum of the n consecutive terms is:

$S_n = u_1 + (u_1+d) + (u_1+2d) + \cdots + (u_n-2d) + (u_n-d) + u_n$. But, reversing them:

$S_n = u_n + (u_n-d) + (u_n-2d) + \cdots + (u_1+2d) + (u_1+d) + u_1$

Adding these two expressions vertically, we get:

$$\begin{array}{rrcccccccccccc} & S_n & = & u_1 & + & (u_1+d) & + & (u_1+2d) & + \cdots + & (u_n-2d) & + & (u_n-d) & + & u_n \\ + & S_n & = & u_n & + & (u_n-d) & + & (u_n-2d) & + \cdots + & (u_1+2d) & + & (u_1+d) & + & u_1 \\ \hline & 2\cdot S_n & = & \multicolumn{11}{l}{\underbrace{(u_1+u_n) + (u_1+u_n) + (u_1+u_n) + \cdots + (u_1+u_n) + (u_1+u_n) + (u_1+u_n)}_{n \text{ of these}}} \end{array}$$

$$2\cdot S_n = n(u_1+u_n) \quad \Rightarrow \quad \boxed{S_n = \frac{1}{2}n(u_1+u_n)}$$

Example 7

Find the sum of the first 50 terms of the sequence 4, 7, 10, 13, ...

The sequence is arithmetic with $u_1 = 4$, $d = 3$ and $n = 50$. Then:

$u_{50} = u_1 + 49d = 4 + 49\cdot 3 = 151$

$S_{50} = \frac{1}{2}\cdot 50(4+151) = 3875$

Exercises - Set C

1. Find the sum of:
   a) the first 20 terms of the sequence $3, 7, 11, 15, \ldots$
   b) the first 22 terms of the sequence $-6, 1, 8, 15, \ldots$
   c) the first 40 terms of the sequence $100, 93, 86, 79, \ldots$

5. Sum of n terms of a geometric sequence

Let us consider the sum of the n first terms of a geometric sequence: $S_n = u_1 + u_2 + \cdots u_{n-1} + u_n$
If we multiply both sides by the common ratio r:

$$r\cdot S_n = u_1\cdot r + u_2\cdot r + \cdots + u_{n-1}\cdot r + u_n\cdot r \quad , \quad \text{that is:}$$
$$r\cdot S_n = u_2 + u_3 + \cdots + u_n + u_n\cdot r$$

Subtracting: $rS_n - S_n = u_n r - u_1 \;\Rightarrow\; S_n(r-1) = u_n r - u_1$

Hence: $\boxed{S_n = \dfrac{u_n\cdot r - u_1}{r-1}}$

Example 8

Find the sum of the first 12 terms of the sequence 2, 6, 18, 54, ...

The sequence is geometric with $u_1 = 2$, $r = 3$ and $n = 12$. Then:

$u_{12} = 2 \cdot 3^{11} = 354\,294$

$$S_{50} = \frac{354\,294 \cdot 3 - 2}{3 - 1} = \frac{1\,062\,880}{2} = 531\,440$$

Exercises - Set D

1. Find the sum of:
   a) the first 10 terms of the sequence $12, 6, 3, 1.5 \ldots$
   b) the first 15 terms of the sequence $6, -3, 1.5, -0.75 \ldots$

Review Exercises

1. The first term of an arithmetic sequence is equal to 6 and the common difference is equal to 3. Find a formula for the nth term and the value of the 50th term.

2. The first term of an arithmetic sequence is equal to 200 and the common difference is equal to -10. Find the value of the 20th term.

3. An arithmetic sequence has a common difference equal to 10 and its 6th term is equal to 52. Find its 15th term.

4. An arithmetic sequence has a its 5th term equal to 22 and its 15th term equal to 62. Find its 100th term.

5. Find the sum of all the integers from 1 to 1000.

6. Find the sum of the first 50 even positive integers.

7. Find the sum of all positive integers, from 5 to 1555 inclusive, that are divisible by 5.

8. Find the terms a_2, a_3, a_4 and a_5 of a geometric sequence if $a_1 = 10$ and the common ratio $r = -1$.

9. Find the 10th term of a geometric sequence if $a_1 = 45$ and the common ration $r = 0.2$.

10. Find a_{20} of a geometric sequence if the first few terms of the sequence are given by $-1/2$, $1/4$, $-1/8$, $1/16$, ...

11. Given the terms $a_{10} = 3/512$ and $a_{15} = 3/16384$ of a geometric sequence, find the exact value of the term a_{30} of the sequence.

12. Find the sum of the first 12 terms of the sequence: 1, 3, 9, 27 ...

13. Find a_{20} given that $a_3 = 1/2$ and $a_5 = 8$.

14. Find a_{30} given that the first few terms of a geometric sequence are given by $-2, 1, -1/2, 1/4 \ldots$

15. Find r given that $a_1 = 10$ and $a_{20} = 10^{-18}$.