Matrices, Determinantes y Sistemas en exámenes BI-NS (extracto)

Mayo 02

(a) Demuestre por inducción matemática que

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix},$$ para todos los valores enteros y positivos de n.

(b) Determine si este resultado es cierto o no para $n = -1$.

a) $M = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$

$\underline{\underline{n=1}}$ $M^1 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^1 & 2^1-1 \\ 0 & 1 \end{pmatrix}$ ✓

$\underline{\underline{n=k}}$ Suponiendo cierto que: $M^k = \begin{pmatrix} 2^k & 2^k-1 \\ 0 & 1 \end{pmatrix}$

vamos a demostrar que: $M^{k+1} = \begin{pmatrix} 2^{k+1} & 2^{k+1}-1 \\ 0 & 1 \end{pmatrix}$

$M^{k+1} = M \cdot M^k = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2^k & 2^k-1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2^k & 2\cdot(2^k-1)+1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 2^{k+1}-2+1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 2^{k+1}-1 \\ 0 & 1 \end{pmatrix}$ ✓

b) $|M| = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$

$M_{11} = 1$, $M_{12} = 0$, $M_{21} = -1$, $M_{22} = 2$

$\rightarrow M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & -1+1/2 \\ 0 & 1 \end{pmatrix} =$

$= \begin{pmatrix} 2^{-1} & 2^{-1}-1 \\ 0 & 1 \end{pmatrix}$ ✓ $\underline{\underline{Sí}}$, también es cierto.

Mayo 04 P3 TZ1 (#7)

Let S be the set of all (2×2) non-singular matrices each of whose elements is either 0 or 1. Two matrices belonging to S are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{and} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(a) Write down the other four members of S.

(b) You are given that S forms a group under matrix multiplication, when the elements of the matrix product are calculated modulo 2.

(i) Find the order of all the members of S whose determinant is negative.

(ii) Hence find a subgroup of S of order 3.

a) $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \{0,1\} \; ; \; \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \right\}$

$\cancel{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$ $\cancel{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}$ $\cancel{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$ $\cancel{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}$ $\cancel{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$ Son todos singulares por tener filas o columnas iguales.

$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ ✓

$\cancel{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}$ $\cancel{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$ $\cancel{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$ $\cancel{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}$ Son todos singulares por tener determinante nulo.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ✓

b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$; $A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $2 \equiv 0 \pmod 2$; $A^3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ $2 \equiv 0 \pmod 2$

$\boxed{\text{orden} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = 3}$

$B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$; $B^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ $2 \equiv 0 \pmod 2$; $B^3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$\boxed{\text{orden} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = 3}$

Nov 05
P3 (#7)

Sea M el conjunto de todas las matrices de la forma $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ donde $x \in \mathbb{R}$.

(a) Compruebe que $(M, +)$ no es un grupo.

(b) Compruebe que M forma un grupo abeliano bajo la multiplicación de matrices. (Puede suponer que la multiplicación de matrices es asociativa).

a) $M = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$

La operación + no está cerrada : $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & x+y \\ 0 & 2 \end{pmatrix} \notin M$

b) Operación cerrada : $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y+x \\ 0 & 1 \end{pmatrix} \in M$ ✓

Elemento neutro : $x = 0 \in \mathbb{R} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M$ ✓

Elemento simétrico : $A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$

$$\left.\begin{array}{l} |A| = 1 \\ \left.\begin{array}{l} A_{11} = 1 \\ A_{12} = 0 \end{array}\right\} \begin{array}{l} A_{21} = -x \\ A_{22} = 1 \end{array} \end{array}\right\} A^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \in M \ ✓$$

Asociativa : Asumimos su cumplimiento en el producto de matrices

Conmutativa : $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y+x \\ 0 & 1 \end{pmatrix}$ ✓

$\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$

Muestra
06 = 08

La matriz cuadrada $\boldsymbol{X}$ es tal que $\boldsymbol{X}^3 = 0$. Compruebe que la inversa de la matriz $\boldsymbol{(I - X)}$ es $\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2$.

$X^3 = 0$

$(I - X) \cdot (I + X + X^2) = I + \cancel{X} + \cancel{X^2} - \cancel{X} - \cancel{X^2} - X^3 = I - 0 = I \Rightarrow (I - X)^{-1} = I + X + X^2$ ✓

Mayo 07

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 6 \\ 4 & 3 \end{pmatrix}$ and $\boldsymbol{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. Given that $\boldsymbol{AX} = k\boldsymbol{X}$, where $k \in \mathbb{R}$, find the values of k for which there is an infinity of solutions for $\boldsymbol{X}$.

$AX = kX \rightarrow AX - kX = 0 \ ; \ (A - kI)X = 0$

Si $\exists (A - kI)^{-1} \longrightarrow X = (A - kI)^{-1} \cdot 0 = 0$ Absurdo. sólo Tendría una solución

Si $\nexists (A - kI)^{-1} \longrightarrow$ Tendrá más de una solución.

Entonces: $|A - kI| = 0 \Rightarrow \begin{vmatrix} 1-k & 6 \\ 4 & 3-k \end{vmatrix} = 0 \ ; \ 3 - k - 3k + k^2 - 24 = 0 \ ;$

$k^2 - 4k - 21 = 0 \ ; \ k = \frac{4 \pm \sqrt{16 + 84}}{2} = \frac{4 \pm 10}{2} \begin{cases} 7 \\ -3 \end{cases}$

Mayo 07
P3 TZ2
(#4)

(a) Show that the set $M=\left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix};\ a,\ b\in\mathbb{R},\ a^2+b^2\neq 0\right\}$ together with matrix multiplication ($\times$) forms a group $\{M,\times\}$.

(b) Find an isomorphism from the multiplicative group of non-zero complex numbers to the group $\{M,\times\}$. Justify your answer.

$M=\left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a,b\in\mathbb{R} \ .\ a^2+b^2\neq 0\right\}$

<u>Operación Cerrada</u>: $\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix}\in M$, $\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix}\in M \Rightarrow \begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix}\cdot\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix}=\begin{pmatrix} a_1a_2-b_1b_2 & -a_1b_2-b_1a_2 \\ b_1a_2+a_1b_2 & -b_1b_2+a_1a_2 \end{pmatrix}$

Comprobamos que el elemento 1,1 es igual que el elemento 2,2 y que los elementos 1,2 y 2,1 son opuestos, luego el producto es una matriz del tipo de las de M.

Por otra parte:

$(a_1a_2-b_1b_2)^2+(b_1a_2+a_1b_2)^2=a_1^2a_2^2-2a_1a_2b_1b_2+b_1^2b_2^2+b_1^2a_2^2+2a_1a_2b_1b_2+$
$+a_1^2b_2^2=a_1^2(a_2^2+b_2^2)+b_1^2(a_2^2+b_2^2)=\underbrace{(a_1^2+b_1^2)}_{\neq 0}\cdot\underbrace{(a_2^2+b_2^2)}_{\neq 0}\neq 0$

Luego la operación está cerrada en M. ✓

<u>E. Neutro</u>: Para $a=1$, $b=0$ $\rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\in M$, ya que $1^2+0^2=1\neq 0$ ✓

<u>Asociativa</u>: No es necesario probar nada especial, ya que el producto de matrices 2x2 cumple la asociativa para cualquier trío de matrices.

<u>E. Simétrico</u>: La inversa de $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ es $\begin{pmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ \frac{-b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix}$ que es del tipo de las matrices pertenecientes a M, y además: $\left(\frac{a}{a^2+b^2}\right)^2+\left(\frac{-b}{a^2+b^2}\right)^2=\frac{a^2}{(a^2+b^2)^2}+\frac{b^2}{(a^2+b^2)^2}=\frac{a^2+b^2}{(a^2+b^2)^2}=\frac{1}{a^2+b^2}\neq 0$

Luego, la inversa también pertenece a M.

Luego $(M,\times)$ es un grupo.

<u>b)</u> $f:\mathbb{C}\longrightarrow M$

Definimos $f(a+bi)=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ que es obviamente biyectiva, y cumple los requisitos para ser homomorfismo:

$f\left[(a_1+b_1i)\cdot(a_2+b_2i)\right]=f\left[(a_1a_2-b_1b_2)+i(a_1b_2+b_1a_2)\right]=\begin{pmatrix} a_1a_2-b_1b_2 & -a_1b_2-b_1a_2 \\ a_1b_2+b_1a_2 & a_1a_2-b_1b_2 \end{pmatrix}$

$f(a_1+b_1i)\cdot f(a_2+b_2i)=\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix}\cdot\begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix}=\begin{pmatrix} a_1a_2-b_1b_2 & -a_1b_2-b_1a_2 \\ b_1a_2+a_1b_2 & -b_1b_2+a_1a_2 \end{pmatrix}$

Vemos que: $\boxed{f\left[(a_1+b_1i)\cdot(a_2+b_2i)\right]=f(a_1+b_1i)\cdot f(a_2+b_2i)}$

Luego f es un isomorfismo.

Muestra 08

Sabiendo que $M = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ y que $M^2 - 6M + kI = 0$, halle k.

$$M = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$$

$$M^2 = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ -18 & 19 \end{pmatrix}$$

$$M^2 - 6M + KI = 0 \Rightarrow \begin{pmatrix} 7 & -6 \\ -18 & 19 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix} + K\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow$$

$\rightarrow$ $7 - 12 + K = 0 \rightarrow K = 5$

$-6 + 6 = 0$ ✓

$-18 + 18 = 0$ ✓

$19 - 24 + K = 0 \rightarrow K = 5$

✓ $\boxed{K = 5}$

Mayo 09 P1 TZ1 #4

Consider the matrix $A = \begin{pmatrix} e^x & e^{-x} \\ 2 + e^x & 1 \end{pmatrix}$, where $x \in \mathbb{R}$.

Find the value of x for which A is singular.

$$|A| = 0 \rightarrow \begin{vmatrix} e^x & e^{-x} \\ 2+e^x & 1 \end{vmatrix} = 0 \quad ; \quad e^x - 2e^{-x} - 1 = 0 \quad ; \quad e^x - \frac{2}{e^x} - 1 = 0 \quad ;$$

$$(e^x)^2 - 2 - e^x = 0 \quad ; \quad e^x = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} \begin{cases} 2 \Rightarrow \boxed{x = \ln 2} \\ -1 \ \times \end{cases}$$

Mayo 10 P2 (TZ1) (#5)

Let A, B and C be non-singular 2×2 matrices, I the 2×2 identity matrix and k a scalar. The following statements are **incorrect**. For each statement, write down the correct version of the right hand side.

(a) $(A+B)^2 = A^2 + 2AB + B^2$

(b) $(A-kI)^3 = A^3 - 3kA^2 + 3k^2A - k^3$

(c) $CA = B \Rightarrow C = \dfrac{B}{A}$

a) $(A+B)^2 = (A+B)\cdot(A+B) = \boxed{A^2 + AB + BA + B^2}$ $AB + BA \neq 2AB$ por no cumplirse la conmutativa

b) $(A-KI)^3 = \boxed{A^3 - 3KA^2 + 3K^2A - K^3I}$ $K^3I \neq K^3$ por tratarse de una operación matricial

c) $CA = B \Rightarrow \boxed{C = B\cdot A^{-1}}$ No existe la 'división' de matrices $\frac{B}{A}$ por no aclararse por qué lado multiplicando la inversa.

Mayo 10 P2 (TZ1) (#2)

The system of equations

$$2x - y + 3z = 2$$
$$3x + y + 2z = -2$$
$$-x + 2y + az = b$$

is known to have more than one solution. Find the value of a and the value of b.

$$\begin{pmatrix} 2 & -1 & 3 \\ 3 & 1 & 2 \\ -1 & 2 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ b \end{pmatrix}$$

$A = \begin{pmatrix} 2 & -1 & 3 \\ 3 & 1 & 2 \\ -1 & 2 & a \end{pmatrix}$ $\quad \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 5 \neq 0 \Rightarrow r(A) \geq 2$

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 2 \\ -1 & 2 & a \end{vmatrix} = 2a + 18 + 2 + 3 + 3a - 8 = 5a + 15$$

$5a + 15 = 0 \Rightarrow a = -3$

- Si $a \neq -3 \Rightarrow \left.\begin{array}{l} r(A) = 3 \\ r(A^*) = 3 \\ n = 3 \end{array}\right\} \Rightarrow$ S. Compatible Determinado (Solución única)

Si $a = -3 \Rightarrow r(A) = 2$

$$A^* = \left(\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 3 & 1 & 2 & -2 \\ -1 & 2 & -3 & b \end{array}\right)$$

$$\begin{vmatrix} 2 & -1 & 2 \\ 3 & 1 & -2 \\ -1 & 2 & b \end{vmatrix} = 2b + 12 - 2 + 2 + 3b + 8 = 5b + 20$$

$5b + 20 = 0 \Rightarrow b = -4$

- Si $\left\{\begin{array}{l} a = -3 \\ b \neq -4 \end{array}\right. \rightarrow \left.\begin{array}{l} r(A) = 2 \\ r(A^*) = 3 \end{array}\right\} \Rightarrow$ S. Incompatible (Sin soluciones)

- Si $\left\{\begin{array}{l} a = -3 \\ b = -4 \end{array}\right. \rightarrow \left.\begin{array}{l} r(A) = 2 \\ r(A^*) = 2 \\ n = 3 \end{array}\right\} \Rightarrow$ S. Compatible Indeterminado (Infinitas soluciones)

Entonces : $\boxed{\begin{array}{l} a = -3 \\ b = -4 \end{array}}$

Mayo 10 P3 (#2)

The relation R is defined for 2×2 matrices such that ARB if and only if there exists a non-singular matrix H such that $AH = HB$.

(a) Show that R is an equivalence relation.

(b) Given that A is singular and ARB, show that B is also singular.

$ARB \Leftrightarrow \exists H$ (no singular) $\mid AH = HB$

a) Reflexiva : $\left.\begin{array}{l} AI = IA \\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ no singular} \end{array}\right\} \Rightarrow ARA$ ✓

Simétrica : $ARB \Rightarrow \exists H$ (no singular) $\mid AH = HB \Rightarrow AHH^{-1} = HBH^{-1} \Rightarrow A = HBH^{-1} \Rightarrow$

$\Rightarrow H^{-1}A = H^{-1}HBH^{-1} \Rightarrow H^{-1}A = BH^{-1} \Rightarrow \left.\begin{array}{l} BH^{-1} = H^{-1}A \\ H^{-1} \text{ no singular} \end{array}\right\} \Rightarrow BRA$ ✓

Transitiva : $\left.\begin{array}{l} ARB \\ BRC \end{array}\right\} \Rightarrow \exists H_1, H_2$ (no singulares) $\left| \begin{array}{l} AH_1 = H_1 B \\ BH_2 = H_2 C \rightarrow B = H_2 C H_2^{-1} \end{array}\right\} \Rightarrow$

$\Rightarrow AH_1 = H_1 H_2 C H_2^{-1} \Rightarrow AH_1 H_2 = H_1 H_2 C H_2^{-1} H_2 \Rightarrow AH_1 H_2 = H_1 H_2 C \Rightarrow ARC$ ✓

Luego $\boxed{R \text{ es una relación de equivalencia}}$

(He aplicado que si H es no singular, H^{-1} también es no singular,
y que $|A^{-1}| = \frac{1}{|A|}$.)

Mayo 11 P1 (TZ2) (#12)

(a) Factorize z^3+1 into a linear and quadratic factor.

Let $\gamma = \dfrac{1+i\sqrt{3}}{2}$.

(b) (i) Show that γ is one of the cube roots of -1.

(ii) Show that $\gamma^2 = \gamma - 1$.

(iii) Hence find the value of $(1-\gamma)^6$.

The matrix A is defined by $A = \begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix}$.

(c) Show that $A^2 - A + I = 0$, where 0 is the zero matrix.

(d) Deduce that

(i) $A^3 = -I$;

(ii) $A^{-1} = I - A$.

a) z^3+1

$$\begin{array}{r|rrrr} & 1 & 0 & 0 & 1 \\ -1 & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

$$\boxed{z^3+1 = (z+1)(z^2-z+1)}$$

b) $\gamma = \dfrac{1+i\sqrt{3}}{2}$

$\gamma^3 = \left(\dfrac{1+i\sqrt{3}}{2}\right)^3 = \dfrac{1}{8}(1+i\sqrt{3})^3 = \dfrac{1}{8}(1+\cancel{3i\sqrt{3}} - 3\cdot 3 - \cancel{3\sqrt{3}i}) = \dfrac{-8}{8} = -1$ ✓

Por lo tanto γ es solución de $z^3+1=0$, por lo tanto $\gamma^2-\gamma+1=0 \Rightarrow$
$\Rightarrow \gamma^2 = \gamma - 1$ ✓

$(1-\gamma)^6 = 1 - 6\gamma + 15\gamma^2 - 20\gamma^3 + 15\gamma^4 - 6\gamma^5 + \gamma^6 =$
$= 1 - 6\gamma + 15(\gamma-1) - 20\cdot(-1) + 15\cdot(-\gamma) - 6\cdot(1-\gamma) + 1 =$
$= 1 - \cancel{6\gamma} + \cancel{15\gamma} - 15 + 20 - \cancel{15\gamma} - 6 + \cancel{6\gamma} + 1 = \boxed{1}$

Sustituimos:
$\gamma^2 = \gamma - 1$
$\gamma^3 = -1$
$\gamma^4 = \gamma^3\gamma = (-1)\cdot\gamma = -\gamma$
$\gamma^5 = \gamma^3\cdot\gamma^2 = (-1)\cdot(\gamma-1) = 1-\gamma$
$\gamma^6 = (\gamma^3)^2 = (-1)^2 = 1$

c) $A = \begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix}$

$A^2 = \begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix}\cdot\begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix} = \begin{pmatrix} \gamma^2 & \gamma + \frac{1}{\gamma} \\ 0 & \frac{1}{\gamma^2} \end{pmatrix} = \begin{pmatrix} \gamma^2 & \frac{\gamma^2+1}{\gamma} \\ 0 & \frac{1}{\gamma^2} \end{pmatrix} = \begin{pmatrix} \gamma-1 & \frac{\gamma-1+1}{\gamma} \\ 0 & \frac{1}{\gamma-1} \end{pmatrix} = \begin{pmatrix} \gamma-1 & 1 \\ 0 & \frac{1}{\gamma-1} \end{pmatrix}$

$A^2 - A + I = \begin{pmatrix} \gamma-1 & 1 \\ 0 & \frac{1}{\gamma-1} \end{pmatrix} - \begin{pmatrix} \gamma & 1 \\ 0 & \frac{1}{\gamma} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cancel{\gamma-1-\gamma+1} & 0 \\ 0 & \frac{1}{\gamma-1} - \frac{1}{\gamma} + 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ✓

$\dfrac{1}{\gamma-1} - \dfrac{1}{\gamma} + 1 = \dfrac{\gamma - (\gamma-1) + \gamma(\gamma-1)}{\gamma(\gamma-1)} = \dfrac{1+\gamma^2-\gamma}{\gamma(\gamma-1)} = \dfrac{0}{\gamma(\gamma-1)} = 0$

d) $A^2 - A + I = 0 \Rightarrow A^2 = A - I$

$A^3 = A\cdot A^2 = A(A-I) = A^2 - A = \cancel{A} - I - \cancel{A} = -I$ ✓

$A^2 - A + I = 0 \Rightarrow A = A^2 + I$

$I = A\cdot A^{-1} = (A^2+I)A^{-1} = A + A^{-1} \Rightarrow \boxed{A^{-1} = I - A}$ ✓

Mayo 11 P2 (TZ2) (#4)

Consider the matrix $A = \begin{pmatrix} \cos 2\theta & \sin\theta \\ -\sin 2\theta & \cos\theta \end{pmatrix}$, for $0 < \theta < 2\pi$.

(a) Show that $\det A = \cos\theta$.

(b) Find the values of θ for which $\det A^2 = \sin\theta$.

a) $|A| = \begin{vmatrix} \cos 2\theta & \sin\theta \\ -\sin 2\theta & \cos\theta \end{vmatrix} = \cos 2\theta \cos\theta + \sin 2\theta \sin\theta = \cos(2\theta - \theta) = \cos\theta$ ✓

b) $|A|^2 = \cos^2\theta$

$\cos^2\theta = \sin\theta$; $1 - \sin^2\theta = \sin\theta$; $0 = \sin^2\theta + \sin\theta - 1$;

$\sin\theta = \dfrac{-1 \pm \sqrt{1+4}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$ → $\sin\theta = \dfrac{-1+\sqrt{5}}{2} \Rightarrow \boxed{\theta = \begin{cases} 38{,}17^\circ \\ 141{,}83^\circ \end{cases}}$

→ $\cancel{\sin\theta = \dfrac{-1-\sqrt{5}}{2}}$

Mayo 12 P1 (TZ2) (#5)

Tres matrices de 2×2 no singulares, $\boldsymbol{A}$, $\boldsymbol{B}$ y $\boldsymbol{X}$, satisfacen la ecuación $4\boldsymbol{A} - 5\boldsymbol{BX} = \boldsymbol{B}$.

(a) Halle $\boldsymbol{X}$ en función de $\boldsymbol{A}$ y $\boldsymbol{B}$.

(b) Sabiendo que $\boldsymbol{A} = 2\boldsymbol{B}$, halle $\boldsymbol{X}$.

a) $4A - 5BX = B$;

$5BX = 4A - B$; $5X = 4AB^{-1} - I$; $\boxed{X = \frac{4}{5}AB^{-1} - \frac{1}{5}I}$

b) $A = 2B \rightarrow X = \frac{4}{5}2BB^{-1} - \frac{1}{5}I = \frac{8}{5}I - \frac{1}{5}I = \boxed{\frac{7}{5}I}$

Mayo 13 P2 (TZ1) (#2)

Find the value of k such that the following system of equations does not have a unique solution.

$$\begin{aligned} kx + y + 2z &= 4 \\ -y + 4z &= 5 \\ 3x + 4y + 2z &= 1 \end{aligned}$$

$A = \begin{pmatrix} k & 1 & 2 \\ 0 & -1 & 4 \\ 3 & 4 & 2 \end{pmatrix}$ $\quad A^* = \left(\begin{array}{ccc|c} k & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 3 & 4 & 2 & 1 \end{array}\right)$

$\begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 10 \neq 0 \Rightarrow r(A) \geq 2$

$\begin{vmatrix} k & 1 & 2 \\ 0 & -1 & 4 \\ 3 & 4 & 2 \end{vmatrix} = -2k + 12 + 6 - 16k = 18 - 18k$, $18 - 18k = 0 \rightarrow k = 1$

• Si $k \neq 1 \Rightarrow \left.\begin{array}{l} r(A) = 3 \\ r(A^*) = 3 \\ n = 3 \end{array}\right| \Rightarrow$ S.C.D.

• Si $k = 1 \Rightarrow r(A) = 2$

$A^* = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -1 & 4 & 5 \\ 3 & 4 & 2 & 1 \end{array}\right)$

$\begin{vmatrix} 1 & 2 & 4 \\ -1 & 4 & 5 \\ 4 & 2 & 1 \end{vmatrix} = 4 - 8 + 40 - 64 + 2 - 10 = 46 - 82 = -36 \neq 0 \Rightarrow r(A^*) = 3$ → Sin solución

$\boxed{\text{No tiene solución única para } k = 1}$

Mayo 13 P2 (TZ2) (#2)

Considere el siguiente sistema de ecuaciones:

$$
\begin{aligned}
0,1x-1,7y+0,9z&=-4,4\\
-2,4x+0,3y+3,2z&=1,2\\
2,5x+0,6y-3,7z&=0,8.
\end{aligned}
$$

(a) Exprese el sistema de ecuaciones en forma matricial.

(b) Halle la solución del sistema de ecuaciones.

a) $\begin{pmatrix} 0{,}1 & -1{,}7 & 0{,}9 \\ -2{,}4 & 0{,}3 & 3{,}2 \\ 2{,}5 & 0{,}6 & -3{,}7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4{,}4 \\ 1{,}2 \\ 0{,}8 \end{pmatrix}$

b) $\boxed{\begin{aligned} x&=-2{,}40 \\ y&=1{,}61 \\ z&=-1{,}57 \end{aligned}}$ Resuelto con Calculadora Gráfica, $x=\dfrac{-932}{389}$ $y=\dfrac{628}{389}$ $z=-\dfrac{612}{389}$ Valores Exactos

Muestra 14 P1 (#7)

Consider the following system of equations:

$$
\begin{aligned}
x+y+z&=1\\
2x+3y+z&=3\\
x+3y-z&=\lambda
\end{aligned}
$$

where $\lambda \in \mathbb{R}$.

(a) Show that this system does not have a unique solution for any value of λ.

(b) (i) Determine the value of λ for which the system is consistent.

(ii) For this value of λ, find the general solution of the system.

$A=\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & -1 \end{pmatrix}$ $A^*=\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 3 \\ 1 & 3 & -1 & \lambda \end{array}\right)$

a) $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}=1\neq 0 \Rightarrow r(A)\geq 2$

$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & -1 \end{vmatrix} = -3+6+1-3+2-3=0 \Rightarrow r(A)=2 \Rightarrow$ El sistema no puede ser compatible determinado para ningún λ

b) $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & 3 & \lambda \end{vmatrix} = 3\lambda+6+3-3-2\lambda-9 = \lambda-3$

$\lambda-3=0$; $\lambda=3$

- Si $\lambda\neq 3 \Rightarrow r(A^*)=3 \Rightarrow$ S. Incompatible
- Si $\lambda=3 \Rightarrow r(A^*)=2 \rightarrow$ S. Compatible Indeterminado

$\left.\begin{aligned} x+y&=1-z \\ 2x+3y&=3-z \end{aligned}\right\}$ $x=\dfrac{\begin{vmatrix} 1-z & 1 \\ 3-z & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}}=\boxed{-2z}$; $y=\dfrac{\begin{vmatrix} 1 & 1-z \\ 2 & 3-z \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}}=\boxed{1+z}$; $\boxed{z\in \mathbb{R}}$