

① a) $A^2 = \begin{pmatrix} 0 & k & t \\ 0 & 0 & k \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & k & t \\ 0 & 0 & k \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & k^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
 $A^4 = A^2 \cdot A^2 = \begin{pmatrix} 0 & 0 & k^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & k^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underline{0}$
 $\boxed{A^{10} = 0} \text{ sol.}$

b) $|B| = \begin{vmatrix} 1 & k & t \\ 0 & 1 & k \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$ B es regular
 $\boxed{B^{-1} = \frac{1}{|B|} \cdot \text{Adj}(B)^t} = \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ k^2-t & -k & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & -k & (k^2-t) \\ 0 & 1 & -k \\ 0 & 0 & 1 \end{bmatrix} \text{ sol.}$

c) $k=0$ $B_0 = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $B^2 = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
 $= \begin{pmatrix} 1 & 0 & 2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$B^3 = \begin{pmatrix} 1 & 0 & 2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$B^4 = B^2 \cdot B^2 = \begin{pmatrix} 1 & 0 & 2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

..... (inducción) ...
 $\boxed{B^{10} = \begin{pmatrix} 1 & 0 & 10t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \text{ sol.}$

④ $\begin{cases} 2A+3B=M & f_1+2f_2 \\ -A+B=N \end{cases} \rightarrow \begin{cases} 5B=M+2N \\ -A+B=N \end{cases} \begin{cases} B=\frac{1}{5}(M+2N) \\ A=B-N \end{cases}$

$\boxed{B = \frac{1}{5} \left[\begin{pmatrix} 8 & 4 & 7 \\ 18 & 11 & -6 \\ 8 & 3 & 13 \end{pmatrix} + 2 \begin{pmatrix} 9 & -2 & 6 \\ 17 & 1 & -10 \\ 9 & 4 & 13 \end{pmatrix} \right]} = \frac{1}{5} \begin{pmatrix} 26 & 0 & 19 \\ 52 & 13 & -26 \\ 26 & 11 & 39 \end{pmatrix}$

$\boxed{A = \frac{1}{5} \begin{pmatrix} 26 & 0 & 19 \\ 52 & 13 & -26 \\ 26 & 11 & 39 \end{pmatrix} - \begin{pmatrix} 9 & -2 & 6 \\ 17 & 1 & -10 \\ 9 & 4 & 13 \end{pmatrix}} = \begin{pmatrix} -\frac{19}{5} & -2 & -\frac{11}{5} \\ -\frac{33}{5} & \frac{8}{5} & \frac{24}{5} \\ -\frac{19}{5} & -\frac{9}{5} & -\frac{26}{5} \end{pmatrix}$
 $= -\frac{1}{5} \begin{pmatrix} 19 & 10 & 11 \\ 33 & -8 & -24 \\ 19 & 9 & 26 \end{pmatrix} \text{ sol.}$

⑤ a) $A^2 = \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix}$; $A^3 = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$
 luego $A^3 = I \Rightarrow A^3 - I = 0$ ged. b) $\boxed{A^{13} = A^{3 \cdot 4 + 1} = A^3 \cdot A^3 \cdot A^3 \cdot A^3 \cdot A = I \cdot I \cdot I \cdot I \cdot A = I \cdot A = A} \text{ sol.}$

② $A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$
 $A^2 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix}$
 $A^3 = \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3a \\ 0 & 1 \end{pmatrix}$

..... (inducción) ...
 $\boxed{A^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}} \text{ sol.}$

$A^{22} - 12A^2 + 2A =$
 $= \begin{pmatrix} 1 & 22a \\ 0 & 1 \end{pmatrix} - 12 \begin{pmatrix} 1 & 2a \\ 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$
 $= \begin{pmatrix} 1 & 22a \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -12 & -24a \\ 0 & -12 \end{pmatrix} + \begin{pmatrix} 2 & 2a \\ 0 & 2 \end{pmatrix}$
 $= \begin{pmatrix} -9 & 0 \\ 0 & -10 \end{pmatrix} \text{ sol.}$

③ $XC + A = C + A^2$
 $XC = C + A^2 - A$
 si existe C^{-1} : $X \underbrace{CC^{-1}} = (A^2 - A + C)C^{-1}$

$\boxed{X = (A^2 - A + C)C^{-1}} \text{ sol.}$

$|C| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 2 \end{vmatrix} = 2 \neq 0 \exists C^{-1}$

$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

luego: $\backslash = A$
 $\boxed{X = (A^2 - A + C)C^{-1} = (A - A + C)C^{-1} = (0 + C)C^{-1} = C \cdot C^{-1} = I_3}$
 $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

→ ⑤ (cont) c) $A^2X + I = A \Rightarrow A(A^2X + I) = AA \Rightarrow A^3X + AI = A^2 \Rightarrow$
 $\Rightarrow IX + A = A^2 \Rightarrow X + A = A^2 \Rightarrow X + A - A = A^2 - A \Rightarrow X + \underline{0} = A^2 - A$
 $\Rightarrow \boxed{X = A^2 - A = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix} - \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 \\ 0 & -1 & -2 \\ -1 & 0 & 1 \end{pmatrix}}$ sol.

⑥ $(A - \lambda I)^2 = B \Rightarrow \lambda?$

λ : "lambda"
 $A - \lambda I = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$
 $= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{pmatrix} =$
 $= \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix}$

$(A - \lambda I)^2 = \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} =$
 $= \begin{pmatrix} \lambda^2+2 & 1-2\lambda & -2\lambda \\ 1-2\lambda & \lambda^2-2\lambda^{(*)} & 1 \\ -2\lambda & 1 & 1+\lambda^2 \end{pmatrix} := B \Rightarrow$

$\begin{matrix} (*) \\ 1+(1-\lambda)^2 \\ = 1+1-2\lambda+\lambda^2 \\ = 2-2\lambda+\lambda^2 \end{matrix} \Rightarrow \begin{cases} \lambda^2+2=6 \Rightarrow \lambda = \pm 2 \quad \checkmark \\ 1-2\lambda=-3 \Rightarrow 2\lambda=4 \quad \boxed{\lambda=2} \\ -2\lambda=-4 \Rightarrow \checkmark \\ 1-2\lambda=-3 \Rightarrow \checkmark \\ 2+\lambda^2-2\lambda=2 \Rightarrow \lambda(\lambda-2)=0 \quad \checkmark \\ \begin{matrix} \lambda=0 \\ \lambda=2 \end{matrix} \\ 1=1 \quad \checkmark \\ -2\lambda=-4 \Rightarrow \boxed{\lambda=+2} \quad \checkmark \\ 1=1 \quad \checkmark \\ 1+\lambda^2=5 \Rightarrow \checkmark \end{cases}$

luego $\boxed{\lambda=2}$ sol.

⑦ a) $A = \begin{pmatrix} m & 0 & 0 \\ 0 & 0 & m \\ 0 & -1 & m+1 \end{pmatrix} \sim \begin{pmatrix} m & 0 & 0 \\ 0 & -1 & m+1 \\ 0 & 0 & m \end{pmatrix} \quad (f_2 \leftrightarrow f_3) \quad (\text{triangular superior}) \Rightarrow$

$\Rightarrow \begin{cases} \text{si } m \neq 0 \Rightarrow \text{rg}(A) = 3 \\ \text{si } m = 0 \Rightarrow \text{rg}(A) = 1 \end{cases} \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}; \quad \underline{\underline{\text{sol}}}$

b) $m = -1$ $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}; \quad XA + A = 2I; \quad XA + IA = 2I; \quad (X+I)A = 2I$

(A es "regular" $\exists \bar{A}^{-1}$) $\Rightarrow (X+I)A\bar{A}^{-1} = 2I\bar{A}^{-1} \Rightarrow (X+I)I = 2\bar{A}^{-1} \Rightarrow X+I = 2\bar{A}^{-1} \Rightarrow X+I - I = 2\bar{A}^{-1} - I$

$\Rightarrow \boxed{X = 2\bar{A}^{-1} - I}$

Cálculo de $\bar{A}^{-1}$: $\bar{A}^{-1} = \frac{1}{|A|} \cdot \text{Adj}(A)^t$

$|A| = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{vmatrix} = - \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -(-1) = 1$

$\text{Adj}(A) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

(matriz "ortogonal")

luego $\bar{A}^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

$\boxed{X = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -1 \end{pmatrix}}$