(1) $z=\sqrt{3}+i$

Opuesto : $\boxed{-z=-\sqrt{3}-i}$

Conjugado : $\boxed{z^*=\sqrt{3}-i}$

Inverso : $\frac{1}{z}=\frac{1}{\sqrt{3}+i}=\frac{\sqrt{3}-i}{(\sqrt{3}+i)(\sqrt{3}-i)}=\frac{\sqrt{3}-i}{3+1}=\boxed{\frac{\sqrt{3}}{4}-i\frac{1}{4}}$

Inverso del conjugado : $\frac{1}{z^*}=\frac{1}{\sqrt{3}-i}=\frac{\sqrt{3}+i}{(\sqrt{3}-i)(\sqrt{3}+i)}=\frac{\sqrt{3}+i}{3+1}=\boxed{\frac{\sqrt{3}}{4}+i\frac{1}{4}}$ De hecho, coincide con el conjugado del inverso.

(2) a)

$123 \div 4$: cociente 30, resto 3 $\rightarrow i^{123}=i^3=\boxed{-i}$

$100 \div 4$: cociente 25, resto 0 $\rightarrow i^{100}=i^0=\boxed{1}$

$33 \div 4$: cociente 8, resto 1 $\rightarrow i^{-33}=i^{-1}=\frac{1}{i}=\frac{-i}{i(-i)}=\frac{-i}{1}=\boxed{-i}$

b)

$44 \div 4$: cociente 11, resto 0 $\rightarrow i^{44}=i^0=1$

$1+i+i^2+i^3+\ldots\ldots+i^{44}=\underbrace{\cancel{1+i-1-i}}+\underbrace{\cancel{1+i-1-i}}+\ldots\underbrace{\cancel{\ldots}}+1=\boxed{1}$

También : $S_{44}=\frac{i^{44}\cdot i-1}{i-1}=\frac{1\cdot i-1}{i-1}=\frac{i-1}{i-1}=1$ ✓

(3) $(2x-3i)(4+i)=8x+2xi-12i-3i^2=(8x+3)+i(2x-12)=$ imaginario puro. $\Rightarrow$

$\Rightarrow 8x+3=0 \rightarrow \boxed{x=-3/8}$

(4) Raíces: $-2,\ -3+2i \longrightarrow -3-2i$

$(z+2)(z-(-3+2i))(z-(-3-2i))=(z+2)(z+3-2i)(z+3+2i)=(z+2)((z+3)^2-4i^2)=$

$=(z+2)(z^2+6z+9+4)=(z+2)(z^2+6z+13)=z^3+6z^2+13z+2z^2+12z+26=\boxed{z^3+8z^2+25z+26}$

$a=8$; $b=25$; $c=26$

(5) $z=a+bi$

$Re(z)=1+Im(z) \Rightarrow a=1+b$ $\Big\}$ $z=(1+b)+bi$

$z^3=\left[(1+b)+bi\right]^3=(1+b)^3+3(1+b)^2bi+3(1+b)(bi)^2+(bi)^3=$

$=1+3b^2+3b+b^3+3bi(1+2b+b^2)+(3+3b)\cdot b^2i^2+b^3i^3=$

$=1+\cancel{3b^2}+3b+b^3+3bi+6b^2i+3b^3i-\cancel{3b^2}-3b^3-b^3i=$

$=(1+3b-2b^3)+i(3b+6b^2+2b^3)=$ nº real $\Rightarrow 2b^3+6b^2+3b=0$

$2b^3+6b^2+3b=0$; $b(2b^2+6b+3)=0$ $\nearrow \boxed{b=0}$ $\searrow 2b^2+6b+3=0$; $b=\frac{-6\pm\sqrt{36-24}}{4}=\frac{-6\pm2\sqrt{3}}{4}=\boxed{\frac{-3\pm\sqrt{3}}{2}}$

$b=0 \rightarrow a=1 \Rightarrow \boxed{z=1}$

$b=\frac{-3+\sqrt{3}}{2} \rightarrow a=1+\frac{-3+\sqrt{3}}{2}=\frac{2-3+\sqrt{3}}{2}=\frac{\sqrt{3}-1}{2} \Rightarrow \boxed{z=\frac{\sqrt{3}-1}{2}+i\,\frac{\sqrt{3}-3}{2}}=0'366-0'634i$

$b=\frac{-3-\sqrt{3}}{2} \rightarrow a=1+\frac{-3-\sqrt{3}}{2}=\frac{2-3-\sqrt{3}}{2}=\frac{-1-\sqrt{3}}{2} \Rightarrow \boxed{z=\frac{-\sqrt{3}-1}{2}+i\,\frac{-\sqrt{3}-3}{2}}=-1'366-2'366i$

(6) Probemos con un ejemplo:

$$z_1 = 2+5i,\quad z_2 = 3-i \;\rightarrow\; z_1 \cdot z_2 = (2+5i)(3-i) = 6-2i+15i-5i^2 = 11+13i$$

$$Re(z_1 \cdot z_2) = 11$$
$$Re(z_1) \cdot Re(z_2) = 2 \cdot 3 = 6$$

Son distintos.

La fórmula correcta sería: $\boxed{Re(z_1 \cdot z_2) = Re(z_1) \cdot Re(z_2) - Im(z_1) \cdot Im(z_2)}$

(7) $9x^2 + 12x + 13 = 0$

$$x = \frac{-12 \pm \sqrt{144-468}}{18} = \frac{-12 \pm 18i}{18} = \boxed{\frac{-2 \pm 3i}{3}}$$

Comprobación:

$$9 \cdot \left(\frac{-2 \pm 3i}{3}\right)^2 + 12 \cdot \frac{-2 \pm 3i}{3} + 13 = \cancel{9}\,\frac{4 \mp 12i + 9i^2}{\cancel{9}} + 4(-2 \pm 3i) + 13 =$$
$$= 4 \mp \cancel{12i} - 9 - 8 \pm \cancel{12i} + 13 = 17 - 17 = 0 \;\checkmark$$

(8) $z^2 = w \rightarrow 4w^2 + 2w - 2 = 0$

$$w = \frac{-2 \pm \sqrt{4+32}}{8} = \frac{-2 \pm 6}{8} = \begin{cases} \frac{1}{2} \Rightarrow z^2 = \frac{1}{2} \rightarrow z = \pm\sqrt{\frac{1}{2}} = \boxed{\frac{\pm\sqrt{2}}{2}} = \pm 0'707 \\ -1 \Rightarrow z^2 = -1 \rightarrow z = \boxed{\pm i} \end{cases}$$

(9)

	4	8	1	-3	-10
1		4	12	13	10
	4	12	13	10	0
-2		-8	-8	-10	
	4	4	5	0	

$$4z^2 + 4z + 5 = 0 \;;\; z = \frac{-4 \pm \sqrt{16-80}}{8} = \frac{-4 \pm \sqrt{-64}}{8} = \frac{-4 \pm 8i}{8} = \frac{-1 \pm 2i}{2} = -\frac{1}{2} \pm i$$

$\boxed{\text{Soluciones: } 1, -2, -\tfrac{1}{2}+i, -\tfrac{1}{2}-i}$

(10) $|z| = 2\sqrt{5}$

$$\frac{25}{z} - \frac{15}{z^*} = 1-8i \;;\; \frac{25z^* - 15z}{z \cdot z^*} = 1-8i \;;\; \frac{25(a-bi) - 15(a+bi)}{(2\sqrt{5})^2} = 1-8i \;;$$

$$25a - 25bi - 15a - 15bi = 20(1-8i) \;;\; 10a - 40bi = 20 - 160i \;\rightarrow\; \begin{cases} 10a = 20 \Rightarrow a = 2 \\ 40b = 160 \Rightarrow b = 4 \end{cases}$$

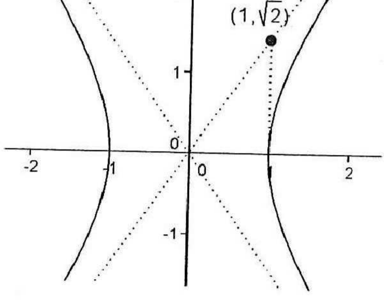
$\boxed{z = 2+4i}$

(11) $z = x + iy \rightarrow z^2 = (x+iy)^2 = x^2 + 2xiy + i^2y^2 = (x^2-y^2) + i2xy$

$$|z|^2 + 3\,Re(z^2) = 4 \rightarrow x^2 + y^2 + 3(x^2 - y^2) = 4 \;;$$

$$x^2 + y^2 + 3x^2 - 3y^2 = 4 \;;\; 4x^2 - 2y^2 = 4 \;;\; \boxed{x^2 - \frac{y^2}{2} = 1}$$

Es una Hipérbola



(12) $1+i\sqrt{3} = \begin{cases} m=\sqrt{1+3}=2 \\ \operatorname{tg}\alpha=\sqrt{3} \rightarrow \alpha=60^\circ \end{cases}$

$(1+i\sqrt{3})^n = \left(2_{60^\circ}\right)^n = 2^n_{n\cdot 60^\circ}$ = nº real , su argumento será $K\cdot 180^\circ$ $(K=0,\pm1,\pm2,\dots)$

$n\cdot 60^\circ = K\cdot 180^\circ \Rightarrow \boxed{n=3K}$ $(K=0,\pm1,\pm2,\dots)$

(13) a) $z^5 = \left(3_{130^\circ}\right)^5 = 243_{650^\circ} = \boxed{243_{290^\circ}} = 243\cdot(\cos 290 + i\operatorname{sen} 290) = \boxed{83'11 - 228'35i}$

b) $z = 3_{130^\circ} = 3(\cos 130 + i\operatorname{sen} 130) = -1'93 + 2'30i$

$w = 2_{312^\circ} = 2(\cos 312 + i\operatorname{sen} 312) = 1'34 - 1'49i$

$z+w = -1'93+2'30i+1'34-1'49i = \boxed{-0'59+0'81i} = \left\{\begin{matrix} m=\sqrt{0'59^2+0'81^2}=1 \\ \operatorname{tg}\alpha = \frac{0'81}{-0'59} \Rightarrow \alpha=126^\circ \end{matrix}\right\} = \boxed{1_{126^\circ}}$

c) $z\cdot w = 3_{130}\cdot 2_{312} = 6_{442^\circ} = \boxed{6_{82^\circ}} = \boxed{0'84+5'94i}$

d) $\frac{z}{w} = \frac{3_{130}}{2_{312}} = 1'5_{-182^\circ} = \boxed{1'5_{178^\circ}} = \boxed{-1'50+0'05i}$

(14) $(z-(3+2i))(z-(3-2i)) = ((z-3)+2i)((z-3)-2i) = (z-3)^2-4i^2 = z^2-6z+9+4 = z^2-6z+13$

$\boxed{z^2-6z+13=0}$

(15) a) $8z^3+27=0$; $z^3 = -\frac{27}{8}$; $z=\sqrt[3]{\frac{-27}{8}}$;

$z = \sqrt[3]{\left(\frac{27}{8}\right)_{180^\circ}} = \left(\sqrt[3]{\frac{27}{8}}\right)_{\frac{180}{3}+\frac{N\cdot 360}{3}} = \left(\frac{3}{2}\right)_{60^\circ+N\cdot 120^\circ} = \begin{cases} \left(\frac{3}{2}\right)_{60^\circ} = \frac{3}{2}\cdot\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = \boxed{\frac{3}{4}+i\frac{3\sqrt{3}}{4}} \\ \left(\frac{3}{2}\right)_{180^\circ} = \boxed{-\frac{3}{2}} \\ \left(\frac{3}{2}\right)_{300^\circ} = \frac{3}{2}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) = \boxed{\frac{3}{4}-i\frac{3\sqrt{3}}{4}} \end{cases}$

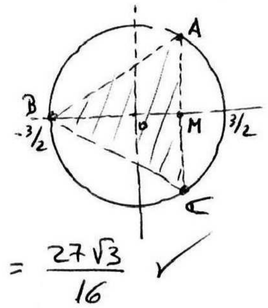
También :

	8	0	0	27
$-3/2$		-12	18	-27
	8	-12	18	0

$\rightarrow z_1 = -3/2$ ✓

$8z^2-12z+18=0$; $4z^2-6z+9=0$; $z = \frac{6\pm\sqrt{36-144}}{8} = \frac{6\pm\sqrt{-108}}{8} = \frac{6\pm 6\sqrt{3}i}{8} = \frac{3\pm 3\sqrt{3}i}{4}$ ✓

b) $AC = 2\cdot\frac{3\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$; $BM = \frac{3}{2}+\frac{3}{4} = \frac{9}{4}$



$\text{Área} = \frac{3\sqrt{3}/2\cdot 9/4}{2} = \frac{27\sqrt{3}}{16}$ ✓

También : $\text{Área} = 3\cdot\text{Área } AOB = 3\cdot\frac{3/2\cdot 3/2\cdot\operatorname{sen}120^\circ}{2} = 3\cdot\frac{3/2\cdot 3/2\cdot\sqrt{3}/2}{2} = \frac{27\sqrt{3}}{16}$ ✓

(16) $z^4+256=0 \rightarrow z^4=-256$; $z=\sqrt[4]{-256}$

$z=\sqrt[4]{-256} = \sqrt[4]{256_{180^\circ}} = \begin{cases} \boxed{4_{45^\circ}} = 4\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) = \boxed{2\sqrt{2}+i2\sqrt{2}} \\ \boxed{4_{135^\circ}} = 4\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right) = \boxed{-2\sqrt{2}+i2\sqrt{2}} \\ \boxed{4_{225^\circ}} = 4\left(-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right) = \boxed{-2\sqrt{2}-i2\sqrt{2}} \\ \boxed{4_{315^\circ}} = 4\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right) = \boxed{2\sqrt{2}-i2\sqrt{2}} \end{cases}$

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$$\sqrt[5]{z} = \begin{cases} -3-4i & \longrightarrow \quad z = (-3-4i)^5 \\ - \\ - \\ - \\ - \end{cases}$$

$$-3-4i = \left\{ \begin{array}{l} m = \sqrt{9+16} = 5 \\ tg\alpha = \frac{4}{3} \Rightarrow \alpha = 233'13° \end{array} \right\} = 5_{233'13} \Rightarrow z = \left(5_{233'13}\right)^5 = \left(3125_{1165'65}\right) = \boxed{3125_{85'65°}}$$

$$\sqrt[5]{z} = \begin{cases} \boxed{5_{233'13°}} \\ 5_{233'13°+72°} = \boxed{5_{305'13°}} \\ 5_{305'13°+72°} = 5_{377'13°} = \boxed{5_{17'13°}} \\ 5_{17,13°+72°} = \boxed{5_{89'13°}} \\ 5_{89'13°+72°} = \boxed{5_{161'13°}} \end{cases}$$

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$$z_1 = r_{\pi/3}$$

$$z_2 = i-1 = \left\{ \begin{array}{l} m = \sqrt{1+1} = \sqrt{2} \\ tg\alpha = \frac{1}{-1} \rightarrow \alpha = \frac{3\pi}{4} \end{array} \right\} = (\sqrt{2})_{3\pi/4}$$

$$|z_1 \cdot z_2^4| = \left| r_{\pi/3} \cdot \left[(\sqrt{2})_{3\pi/4}\right]^4 \right| = \left| r_{\pi/3} \cdot 4_{3\pi} \right| = \left| (4r)_{\frac{10\pi}{3}} \right| = 4r \; ; \; 4r = 12 \Rightarrow \boxed{r=3}$$

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$$\sqrt[5]{-i} = \sqrt[5]{1_{270°}} = \begin{cases} 1_{54} = \boxed{0'59 + 0'81i} \\ 1_{126} = \boxed{-0'59 + 0'81i} \\ 1_{198} = \boxed{-0'95 - 0'31i} \\ 1_{270} = \boxed{-i} \\ 1_{342} = \boxed{0'95 - 0'31i} \end{cases}$$

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$$\left. \begin{array}{l} 2z_1 - iz_2 = 1 \\ z_1 - z_2 = 1-2i \end{array} \right\} \begin{array}{l} \longrightarrow \\ \times(-2) \longrightarrow \end{array} \begin{array}{r} 2z_1 - iz_2 = 1 \\ -2z_1 + 2z_2 = -2+4i \\ \hline (2-i)z_2 = -1+4i \end{array}$$

$$z_2 = \frac{-1+4i}{2-i} = \frac{(-1+4i)(2+i)}{(2-i)(2+i)} = \frac{-2-i+8i+4i^2}{4-i^2} = \boxed{-\frac{6}{5}+\frac{7}{5}i}$$

$\times(-i)$:

$$\begin{array}{r} 2z_1 - iz_2 = 1 \\ -iz_1 + iz_2 = -i-2 \\ \hline (2-i)z_1 \quad = -1-i \end{array}$$

$$z_1 = \frac{3-i}{2-i} = \frac{(-1-i)(2+i)}{(2-i)(2+i)} = \frac{-2-i-2i-i^2}{4-i^2} = \boxed{\frac{-1}{5} - \frac{3}{5}i}$$

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a) $$\frac{u}{v} = \frac{1+i\sqrt{3}}{1+i} = \frac{(1+i\sqrt{3})(1-i)}{(1+i)(1-i)} = \frac{1-i+i\sqrt{3}+\sqrt{3}}{1+1} = \frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}i \quad ✓$$

b) $$u = 1+i\sqrt{3} = \left\{ \begin{array}{l} m = \sqrt{1+3} = 2 \\ tg\alpha = \frac{\sqrt{3}}{1} \rightarrow \alpha = \frac{\pi}{3} \end{array} \right\} = 2_{\pi/3}$$

$$v = 1+i = \left\{ \begin{array}{l} m = \sqrt{1+1} = \sqrt{2} \\ tg\alpha = \frac{1}{1} = 1 \rightarrow \alpha = \frac{\pi}{4} \end{array} \right\} = (\sqrt{2})_{\pi/4}$$

$$\frac{u}{v} = \frac{2_{\pi/3}}{(\sqrt{2})_{\pi/4}} = \left(\frac{2}{\sqrt{2}}\right)_{\frac{\pi}{3}-\frac{\pi}{4}} = (\sqrt{2})_{\pi/12} = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \quad ✓$$

c) $\frac{u}{v} = \frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2} i = \sqrt{2}\cos\frac{\pi}{12} + \sqrt{2}\operatorname{sen}\frac{\pi}{12} i$

$$\left.\begin{array}{l}\operatorname{sen}\pi/12 = \dfrac{\sqrt{3}-1}{2\sqrt{2}}\\ \cos\pi/12 = \dfrac{\sqrt{3}+1}{2\sqrt{2}}\end{array}\right\} \Rightarrow \operatorname{tg}\frac{\pi}{12} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{3-2\sqrt{3}+1}{3-1} = \frac{4-2\sqrt{3}}{2} = \boxed{2-\sqrt{3}}$$

$a=2;\ b=-1$

(22) $1+\frac{1}{3}e^{i\theta} + \frac{1}{9}e^{2i\theta} + \cdots$

a) $z = \frac{1/3\, e^{i\theta}}{1} = \frac{1}{3}e^{i\theta}$

$|z| = \left|\frac{1}{3}e^{i\theta}\right| = \left|\frac{1}{3}\cos\theta + i\frac{1}{3}\operatorname{sen}\theta\right| = \sqrt{\frac{\cos^2\theta}{9} + \frac{\operatorname{sen}^2\theta}{9}} = \sqrt{\frac{1}{9}} = \frac{1}{3} < 1$ ✓

b) $S_\infty = \frac{a_1}{1-r} = \frac{1}{1-\frac{1}{3}e^{i\theta}} = \frac{3}{3-e^{i\theta}} = \frac{3}{(3-\cos\theta)-i\operatorname{sen}\theta} = \frac{3\cdot[(3-\cos\theta)+i\operatorname{sen}\theta]}{[(3-\cos\theta)-i\operatorname{sen}\theta]\cdot[(3-\cos\theta)+i\operatorname{sen}\theta]} =$

$= \frac{(9-3\cos\theta)+i\,3\operatorname{sen}\theta}{(3-\cos\theta)^2-(i\operatorname{sen}\theta)^2} = \frac{(9-3\cos\theta)+i\,3\operatorname{sen}\theta}{9-6\cos\theta+\cos^2\theta+\operatorname{sen}^2\theta} = \boxed{\frac{9-3\cos\theta}{10-6\cos\theta} + i\frac{3\operatorname{sen}\theta}{10-6\cos\theta}}$

c) $1+\frac{1}{3}e^{i\theta} + \frac{1}{9}e^{2i\theta}+\cdots = 1+\frac{1}{3}\cos\theta + \frac{1}{3}i\operatorname{sen}\theta + \frac{1}{9}\cos 2\theta + \frac{1}{9}i\operatorname{sen}2\theta + \cdots =$

$= \left(1+\frac{1}{3}\cos\theta + \frac{1}{9}\cos 2\theta + \frac{1}{27}\cos 3\theta + \cdots\right) + i\left[\frac{1}{3}\operatorname{sen}\theta + \frac{1}{9}\operatorname{sen}2\theta + \frac{1}{27}\operatorname{sen}3\theta + \cdots\right]$

Identificando con el resultado del apartado (b):

$1+\frac{1}{3}\cos\theta + \frac{1}{9}\cos 2\theta + \cdots = \frac{9-3\cos\theta}{10-6\cos\theta}$

$\frac{1}{3}\operatorname{sen}\theta + \frac{1}{9}\operatorname{sen}2\theta + \frac{1}{27}\operatorname{sen}3\theta + \cdots = \frac{3\operatorname{sen}\theta}{10-6\cos\theta} \xRightarrow{(\times 3)}$

$\Rightarrow \operatorname{sen}\theta + \frac{1}{3}\operatorname{sen}2\theta + \frac{1}{9}\operatorname{sen}3\theta + \cdots = \frac{9\operatorname{sen}\theta}{10-6\cos\theta}$ ✓

(23) $z = \cos\theta + i\operatorname{sen}\theta$

a) $z^3 = (\cos\theta + i\operatorname{sen}\theta)^3 = \cos^3\theta + 3\cos^2\theta\, i\operatorname{sen}\theta + 3\cos\theta\,(i\operatorname{sen}\theta)^2 + (i\operatorname{sen}^3\theta) =$

$= [\cos^3\theta - 3\cos\theta\operatorname{sen}^2\theta] + i[3\cos^2\theta\operatorname{sen}\theta - \operatorname{sen}^3\theta] =$

$= [\cos^3\theta - 3\cos\theta(1-\cos^2\theta)] + i[3(1-\operatorname{sen}^2\theta)\operatorname{sen}\theta - \operatorname{sen}^3\theta] =$

$= [4\cos^3\theta - 3\cos\theta] + i[3\operatorname{sen}\theta - 4\operatorname{sen}^3\theta]$

b) $z = 1_\theta \rightarrow z^3 = (1_\theta)^3 = (1^3)_{3\theta} = 1_{3\theta} = \cos 3\theta + i\operatorname{sen}3\theta \Rightarrow$

$$\Rightarrow \boxed{\begin{array}{l}\cos 3\theta = 4\cos^3\theta - 3\cos\theta\\ \operatorname{sen}3\theta = 3\operatorname{sen}\theta - 4\operatorname{sen}^3\theta\end{array}}$$ ✓