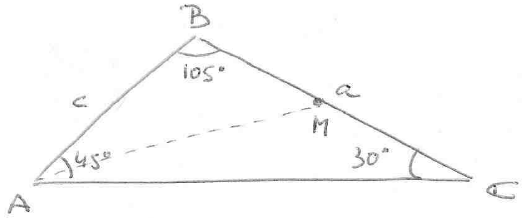
(1)

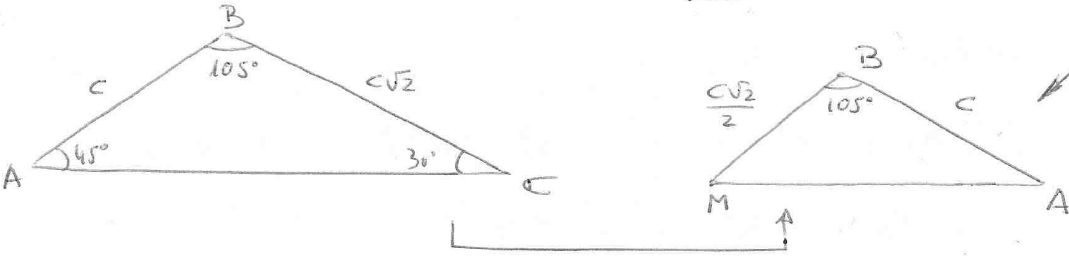


$$\frac{a}{\operatorname{sen}45^\circ} = \frac{c}{\operatorname{sen}30^\circ}$$

$$\frac{a}{\sqrt{2}/2} = \frac{c}{1/2}$$

$$\boxed{a = c\cdot\sqrt{2}}$$

He girado el triángulo para facilitar la visión de semejanza



Son semejantes ya que $\hat{B} = 105^\circ$ es común

y $\frac{AB}{MB} = \frac{c}{c\sqrt{2}/2} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\frac{BC}{BA} = \frac{c\sqrt{2}}{c} = \sqrt{2}$ razón de semejanza

Por lo tanto: $\boxed{B\hat{M}A = B\hat{A}C = 45^\circ}$

$$\left.\begin{array}{l} BC = \sqrt{2}\cdot BA \\ AC = \sqrt{2}\cdot MA \end{array}\right\} \longrightarrow BC\cdot AC = \sqrt{2}\,BA\cdot\sqrt{2}\,MA = \boxed{2\cdot BA\cdot MA}$$

(2) Para obtener 9 bolas blancas, deberíamos no haber encontrado en la colocación inmediatamente anterior dos bolas del mismo color, y esto es imposible con 4 bolas negras y 5 blancas.

(3) $3^n + 5^n = K\cdot(3^{n-1} + 5^{n-1}) \quad (K \in \mathbb{N})$

$$3^n + 5^n = K\cdot 3^{n-1} + K\cdot 5^{n-1}$$

$$3^n - K\cdot 3^{n-1} = K\cdot 5^{n-1} - 5^n$$

$$3^{n-1}\cdot(3-K) = 5^{n-1}\cdot(K-5)$$

Al tener el mismo signo el 1º y 2º miembros:

signo $(3-K) =$ signo $(K-5) = +$ $\Rightarrow \left.\begin{array}{l} K\leq 3 \\ K\geq 5\end{array}\right\}$ Sin solución

signo $(3-K) =$ signo $(K-5) = -$ $\Rightarrow \left.\begin{array}{l} K\geq 3 \\ K\leq 5\end{array}\right\} \Rightarrow K=3,\ K=4,\ K=5$

- $K=3 \Rightarrow 3^{n-1}\cdot 0 = 5^{n-1}\cdot(-2) \Rightarrow -2\cdot 5^{n-1} = 0$ Sin solución
- $K=5 \Rightarrow 3^{n-1}\cdot(-2) = 5^{n-1}\cdot 0 \Rightarrow -2\cdot 3^{n-1} = 0$ Sin solución
- $K=4 \Rightarrow 3^{n-1}\cdot(-1) = 5^{n-1}\cdot(-1) \Rightarrow 3^{n-1} = 5^{n-1} \Rightarrow \left(\frac{3}{5}\right)^{n-1} = 0 \Rightarrow n-1=0 \Rightarrow$

$\Rightarrow \boxed{n=1}$

④ $P(x) = 3x^2 + 3mx + m^2 - 1$

$$x = \frac{-3m \pm \sqrt{9m^2 - 12(m^2-1)}}{2} = \frac{-3m \pm \sqrt{12-3m^2}}{6}$$

$$x_1 = \frac{-3m + \sqrt{12-3m^2}}{6} \qquad x_2 = \frac{-3m - \sqrt{12-3m^2}}{6}$$

Veamos qué se deduce de la tesis:

$$P(x_1^3) = P(x_2^3) \iff 3x_1^6 + 3mx_1^3 + \cancel{m^2-1} = 3x_2^6 + 3mx_2^3 + \cancel{m^2-1} \iff$$

$$\iff x_1^6 - x_2^6 = m x_2^3 - m x_1^3 \iff$$

$$\iff (x_1^3 + x_2^3)\cdot\cancel{(x_1^3 - x_2^3)} = -m\,\cancel{(x_1^3 - x_2^3)} \quad \overset{(* \text{ si } x_1 \neq x_2)}{\iff}$$

$$\iff x_1^3 + x_2^3 = -m$$

Veamos entonces que $x_1^3 + x_2^3 = -m$:

$$x_1^3 + x_2^3 = \frac{(-3m + \sqrt{12-3m^2})^3}{6^3} + \frac{(-3m - \sqrt{12-3m^2})^3}{6^3} =$$

$$= \frac{-27m^3 + \cancel{27m^2\sqrt{12-3m^2}} - 9m(12-3m^2) + \cancel{(\sqrt{12-3m^2})^3} - 27m^3 - \cancel{27m^2\sqrt{12-3m^2}} - 9m(12-3m^2) - \cancel{(\sqrt{12-3m^2})^3}}{216} =$$

$$= \frac{-54m^3 - 18m(12-3m^2)}{216} = \frac{-\cancel{54m^3} - 216m + \cancel{54m^3}}{216} = -m \quad ✓$$

(*) Si $x_1 = x_2 \Rightarrow 12 - 3m^2 = 0 \Rightarrow m = \begin{cases} +2 \Rightarrow x_1 = x_2 = -1 \\ -2 \Rightarrow x_1 = x_2 = +1 \end{cases}$

En los dos casos, $x_1^3 = x_1$, $x_2^3 = x_2$ por lo que es obvia la demostración.

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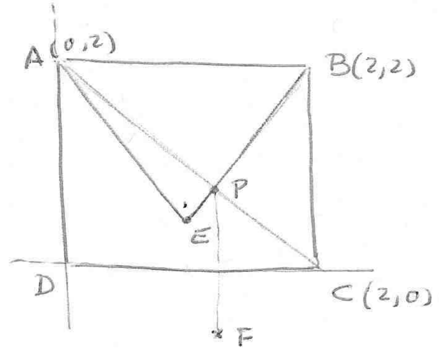

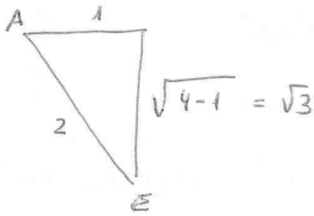

$\boxed{E(1, 2-\sqrt{3})}$

Ecuación $\overline{EB}$: $\vec{EB} = (2-1, 2-2+\sqrt{3}) = (1, \sqrt{3})$ pendiente $=\sqrt{3}$

$y-2 = \sqrt{3}(x-2)$ $\boxed{y = \sqrt{3}\,x + 2(1-\sqrt{3})}$

Ecuación $\overline{AC}$: $\vec{AC} = (2,-2)$ pendiente $= -1$

$y-0 = -1(x-2)$ $\boxed{y = 2-x}$

Punto P : $\left.\begin{array}{l} y = \sqrt{3}\,x + 2(1-\sqrt{3}) \\ y = 2-x \end{array}\right\}$

$\sqrt{3}\,x + 2 - 2\sqrt{3} = 2 - x$

$(1+\sqrt{3})x = 2\sqrt{3}$

$x = \dfrac{2\sqrt{3}}{1+\sqrt{3}} = \dfrac{2\sqrt{3}(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})} = \dfrac{2\sqrt{3}-6}{1-3} = \dfrac{2\sqrt{3}-6}{-2} = \boxed{3-\sqrt{3}}$

$x = 3-\sqrt{3} \Rightarrow y = 2-3+\sqrt{3} = \boxed{\sqrt{3}-1}$

$\boxed{P(3-\sqrt{3}, \sqrt{3}-1)}$

Punto F : $\boxed{F(3-\sqrt{3}, 1-\sqrt{3})}$

a) $C(2,0)$, $E(1, 2-\sqrt{3})$, $F(3-\sqrt{3}, 1-\sqrt{3})$

$\vec{CE} = (-1, 2-\sqrt{3})$ $|\vec{CE}| = \sqrt{1+(2-\sqrt{3})^2} = \sqrt{1+4-4\sqrt{3}+3} = \sqrt{8-4\sqrt{3}}$ ✓

$\vec{CF} = (1-\sqrt{3}, 1-\sqrt{3})$ $|\vec{CF}| = \sqrt{(1-\sqrt{3})^2+(1-\sqrt{3})^2} = \sqrt{1-2\sqrt{3}+3+1-2\sqrt{3}+3} = \sqrt{8-4\sqrt{3}}$ ✓

$\vec{EF} = (2-\sqrt{3}, -1)$ $|\vec{EF}| = \sqrt{(2-\sqrt{3})^2+1} = \sqrt{4-4\sqrt{3}+3+1} = \sqrt{8-4\sqrt{3}}$ ✓

b) $D(0,0)$, $E(1, 2-\sqrt{3})$, $F(3-\sqrt{3}, 1-\sqrt{3})$

$\vec{ED} = (-1, \sqrt{3}-2)$ $\vec{ED}\cdot\vec{EF} = -1(2-\sqrt{3}) + (\sqrt{3}-2)\cdot(-1) =$

$\vec{EF} = (2-\sqrt{3}, -1)$ $= -2+\sqrt{3}-\sqrt{3}+2 = 0 \Rightarrow D\hat{E}F = 90^\circ$ ✓

$|\vec{ED}| = \sqrt{1+(\sqrt{3}-2)^2} = \sqrt{1+3-4\sqrt{3}+4} = \sqrt{8-4\sqrt{3}}$ ✓

$|\vec{EF}| = \sqrt{8-4\sqrt{3}}$ ✓

c) $B(2,2)$, $D(0,0)$, $F(3-\sqrt{3}, 1-\sqrt{3})$

$\vec{BD} = (-2,-2)$ $|\vec{BD}| = \sqrt{4+4} = \sqrt{8}$ ✓

$\vec{BF} = (1-\sqrt{3}, -1-\sqrt{3})$ $|\vec{BF}| = \sqrt{(1-\sqrt{3})^2+(-1-\sqrt{3})^2} = \sqrt{1-2\sqrt{3}+3+1+2\sqrt{3}+3} = \sqrt{8}$ ✓

d) $P(3-\sqrt{3}, \sqrt{3}-1)$, $D(0,0)$, $F(3-\sqrt{3}, 1-\sqrt{3})$

$\vec{PD} = (\sqrt{3}-3, 1-\sqrt{3})$ $|\vec{PD}| = \sqrt{(\sqrt{3}-3)^2+(1-\sqrt{3})^2} = \sqrt{3-6\sqrt{3}+9+1-2\sqrt{3}+3} = \sqrt{16-8\sqrt{3}}$ ✓

$\vec{PF} = (0, 2-2\sqrt{3})$ $|\vec{PF}| = \sqrt{(2-2\sqrt{3})^2} = \sqrt{4-8\sqrt{3}+12} = \sqrt{16-8\sqrt{3}}$ ✓

$\vec{DF} = (3-\sqrt{3}, 1-\sqrt{3})$ $|\vec{DF}| = \sqrt{(3-\sqrt{3})^2+(1-\sqrt{3})^2} = \sqrt{3-6\sqrt{3}+9+1-2\sqrt{3}+3} = \sqrt{16-8\sqrt{3}}$ ✓

(6) $x^2 f(x) + f(1-x) = 2x - x^4$

Sustituyendo x por $1-x$:

$$(1-x)^2 f(1-x) + f(1-(1-x)) = 2(1-x) - (1-x)^4$$

$$(1-x)^2 f(1-x) + f(x) = 2(1-x) - (1-x)^4$$

$$\left.\begin{array}{l} x^2 f(x) + f(1-x) = 2x - x^4 \\ f(x) + (1-x)^2 f(1-x) = 2(1-x) - (1-x)^4 \end{array}\right\}$$

$$x^2(1-x)^2 f(x) + (1-x)^2 f(1-x) = 2x(1-x)^2 - (1-x)^2 x^4$$

$$-\ f(x) \quad - (1-x)^2 f(1-x) = -2(1-x) + (1-x)^4$$

$$\left[x^2(1-x)^2 - 1\right] f(x) = 2x(1-x)^2 - (1-x)^2 x^4 - 2(1-x) + (1-x)^4$$

$$\boxed{f(x) = \frac{2x(1-x)^2 - (1-x)^2 x^4 - 2(1-x) + (1-x^4)}{x^2(1-x)^2 - 1}}$$