Solving Exponential & Logarithmic Equations

- **Properties of Exponential and Logarithmic Equations**

  Let a be a positive real number such that $a \neq 1$, and let x and y be real numbers. Then the following properties are true:

  1. $a^x = a^y$ if and only if $x = y$
  2. $\log_a x = \log_a y$ if and only if $x = y$ $(x > 0,\ y > 0)$

- **Inverse Properties of Exponents and Logarithms**

	Base a	*Natural Base e*
1.	$\log_a(a^x) = x$	$\ln(e^x) = x$
2.	$a^{(\log_a x)} = x$	$e^{(\ln x)} = x$

- **Solving Exponential and Logarithmic Equations**

  1. To solve an exponential equation, first isolate the exponential expression, then **take the logarithm of both sides of the equation** and solve for the variable.
  2. To solve a logarithmic equation, first isolate the logarithmic expression, then **exponentiate both sides of the equation** and solve for the variable.

     For Instance: If you wish to solve the equation, $\ln x = 2$, you **exponentiate** both sides of the equation to solve it as follows:

$\ln x = 2$	Original equation
$e^{\ln x} = e^2$	Exponentiate both sides
$x = e^2$	Inverse property

Or you can simply rewrite the logarithmic equation in exponential form to solve (i.e. $\ln x = 2$ if and only if $e^2 = x$).

Note: You should *always* check your solution in the original equation.

Example 1:

Solve each equation.

a. $4^{x+2} = 64$

b. $\ln(2x - 3) = \ln 11$

Solution:

a.

$4^{x+2} = 64$	Original Equation
$4^{x+2} = 4^3$	Rewrite with like bases
$x + 2 = 3$	Property of exponential equations
$x = 1$	Subtract 2 from both sides

The solution is 1. Check this in the original equation.

b.

$\ln(2x - 3) = \ln 11$	Original Equation
$2x - 3 = 11$	Property of logarithmic equations
$2x = 14$	Add 3 to both sides
$x = 7$	Divide both sides by 2

The solution is 7. Check this in the original equation.

Example 2:

Solve $5 + e^{x+1} = 20$.

Solution:

$5 + e^{x+1} = 20$	Original Equation
$e^{x+1} = 15$	Subtract 5 from both sides
$\ln e^{x+1} = \ln 15$	Take the logarithm of both sides
$x + 1 = \ln 15$	Inverse Property
$x = -1 + \ln 15 \approx 1.708$	Subtract 1 from both sides

Check:

$5 + e^{x+1} = 20$	Original Equation
$5 + e^{1.708+1} \stackrel{?}{=} 20$	Substitute 1.708 for x
$5 + e^{2.708} \stackrel{?}{=} 20$	Simplify
$5 + 14.999 \approx 20$	Solution checks ✓

Example 3:

Solve the exponential equations.

a. $2^x = 7$ b. $4^{x-3} = 9$ c. $2e^x = 10$

Solutions:

Method 1:

a. $2^x = 7$ Original Equation

$\log 2^x = \log 7$ Take the logarithm of both sides

$x(\log 2) = \log 7$ Property of Logarithms

$x = \frac{\log 7}{\log 2} \approx 2.807$ Solve for x

Method 2:

a. $2^x = 7$ Original Equation

$\log_2 2^x = \log_2 7$ Take the logarithm of both sides

$x = \log_2 7$ Inverse Property

$x = \frac{\log 7}{\log 2} \approx 2.807$ Change of Base Formula

Method 1:

b. $4^{x-3} = 9$ Original Equation

$\log 4^{x-3} = \log 9$ Take the logarithm of both sides

$(x-3)\log 4 = \log 9$ Property of Logarithms

$x - 3 = \frac{\log 9}{\log 4}$ Divide both sides by $\log 4$

$x = 3 + \frac{\log 9}{\log 4} \approx 4.585$ Solve for x

Method 2:

b. $4^{x-3} = 9$ Original Equation

$\log_4 4^{x-3} = \log_4 9$ Take the logarithm of both sides

$x - 3 = \log_4 9$ Inverse Property

$x - 3 = \frac{\log 9}{\log 4}$ Change of Base Formula

$x = 3 + \frac{\log 9}{\log 4} \approx 4.585$ Solve for x

c. $2e^x = 10$ Original Equation

$e^x = 5$ Divide both sides by 2

$\ln e^x = \ln 5$ Take the logarithm of both sides

$x = \ln 5 \approx 1.609$ Inverse Property

Example 4:

Solve $2\log_4 x = 5$.

Solution:

$2\log_4 x = 5$ Original Equation

$\log_4 x = \frac{5}{2}$ Divide both sides by 2

$4^{5/2} = x$ Change to exponential form

$x = 32$ Simplify

Example 5:

Solve $3\log x = 6$.

Solution:

$3\log x = 6$ Original Equation

$\log x = 2$ Divide both sides by 3

$10^2 = x$ Change to exponential form

$x = 100$ Simplify

Example 6:

Solve $20\ln 0.2x = 30$.

Solution:

$20\ln 0.2x = 30$ Original Equation

$\ln 0.2x = 1.5$ Divide both sides by 20

$0.2x = e^{1.5}$ Change to exponential form

$x = 5e^{1.5} \approx 22.408$ Divide both sides by 0.2

Example 7: Solving a Logarithmic Equation using Exponentiation

Solve $\log_3 2x - \log_3(x-3) = 1$

Solution:

$\log_3 2x - \log_3(x-3) = 1$ Original Equation

$\log_3 \frac{2x}{x-3} = 1$ Condense the left side

$3^{\log_3 \frac{2x}{x-3}} = 3^1$ Exponentiate both sides

$\frac{2x}{x-3} = 3$ Inverse Property

$2x = 3x - 9$ Multiply both sides by $x - 3$

$x = 9$ Solve for x

Practice Problems

Solve the following equations:

Remember that the arguments of all logarithms must be greater than 0. Also exponentials in the form of a^x will be greater than 0. Be sure to check all your answers in the original equation.

1. $3^{x-1} = 81$
2. $8^x = 4$
3. $e^x = 5$
4. $-14 + 3e^x = 11$
5. $-6 + \ln 3x = 0$
6. $\log(3x + 1) = 2$
7. $\ln x - \ln 3 = 4$
8. $2\ln 3x = 4$
9. $5^{x+2} = 4$
10. $\ln(x + 2)^2 = 6$
11. $4^{-3x} = 0.25$
12. $2e^{2x} - 5e^x - 3 = 0$
13. $\log_7 3 + \log_7 x = \log_7 32$
14. $2\log_6 4x = 0$
15. $\log_2 x + \log_2(x - 3) = 2$
16. $\log_2(x + 5) - \log_2(x - 2) = 3$
17. $4\ln(2x + 3) = 11$
18. $\log x - \log 6 = 2\log 4$
19. $2^x = 64$
20. $5^x = 25$
21. $4^{x-3} = \frac{1}{16}$
22. $3^{x-2} = 81$
23. $\log_3 x = 5$
24. $\log_4 x = 3$
25. $\log_2 2x = \log_2 100$
26. $\ln(x + 4) = \ln 7$
27. $\log_3(2x + 1) = 2$
28. $\log_5(x - 10) = 2$
29. $3^x = 500$
30. $8^x = 1000$
31. $\ln x = 7.25$
32. $\ln x = -0.5$
33. $2e^{0.5x} = 45$
34. $100e^{-0.6x} = 20$
35. $12(1 - 4^x) = 18$
36. $25(1 - e^t) = 12$
37. $\log 2x = 1.5$
38. $\log_2 2x = -0.65$
39. $\frac{1}{3}\log_2 x + 5 = 7$
40. $4\log_5(x + 1) = 4.8$
41. $\log_2 x + \log_2 3 = 3$
42. $2\log_4 x - \log_4(x - 1) = 1$

Practice Problems Answers

1. 5
2. $\frac{2}{3}$
3. 1.609
4. 2.120
5. 134.476
6. 33
7. 163.794
8. 2.463
9. -1.139
10. 18.086, -22.086
11. $\frac{1}{3}$
12. 1.099
13. $\frac{32}{3}$
14. $\frac{1}{4}$
15. 4
16. 3
17. 6.321
18. 96
19. 6
20. 2
21. 1
22. 6
23. 243
24. 64
25. 50
26. 3
27. 4
28. 35
29. 5.66
30. 3.32
31. 1408.10
32. 0.61
33. 6.23
34. 2.68
35. No Solution
36. –0.65
37. 15.81
38. 0.32
39. 64
40. 5.90
41. $\frac{8}{3}$
42. 2