1) Synthetic Division: The Process

(Ruffini's rule)

2) Remainder Theorem
3) Factor Theorem

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1) Synthetic division. Ruffini's rule

Synthetic division (Ruffini's rule) is a shorthand, or shortcut, method of polynomial division in the special case of dividing by a linear factor *(x – r)* and it *only* works in this case. Synthetic division is generally used, however, not for dividing out factors but for finding zeroes (or roots) of polynomials.

In mathematics, **Ruffini's rule** allows the rapid division of any polynomial by a binomial of the form $x - r$. Ruffini's rule is a special case of synthetic division when the divisor is a linear factor.

Polynomial division by $x - r$

A worked example of polynomial division.

Let:

$$P(x) = 2x^3 + 3x^2 - 4$$
$$Q(x) = x + 1.$$

We want to divide $P(x)$ by $Q(x)$ using Ruffini's rule. The main problem is that $Q(x)$ is not a binomial of the form $x - r$, but rather $x + r$. We must rewrite $Q(x)$ in this way:

$$Q(x) = x + 1 = x - (-1).$$

Now we apply the algorithm:

<table>
<tr>
<td>1. Write down the coefficients and r. Note that, as P(x) didn't contain a coefficient for x, we've written 0:</td>
<td><pre>
    | 2   3   0   -4
    |
 -1 |
----|----------------------------
    |
    |
</pre></td>
</tr>
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2. Pass the first coefficient down:	$\begin{array}{r\|rrrr} & 2 & 3 & 0 & -4 \\ -1 & & & & \\ \hline & 2 & & & \end{array}$
3. Multiply the last obtained value by *r*:	$\begin{array}{r\|rrrr} & 2 & 3 & 0 & -4 \\ -1 & & -2 & & \\ \hline & 2 & & & \end{array}$
4. Add the values:	$\begin{array}{r\|rrrr} & 2 & 3 & 0 & -4 \\ -1 & & -2 & & \\ \hline & 2 & 1 & & \end{array}$
5. Repeat steps 3 and 4 until we've finished:	$\begin{array}{r\|rrrr} & 2 & 3 & 0 & -4 \\ -1 & & -2 & -1 & 1 \\ \hline & 2 & 1 & -1 & -3 \end{array}$ (result coefficients)(remainder)

So, if *original number* = *divisor* × *quotient* + *remainder*, then

$P(x) = Q(x)R(x) + s$, where

$R(x) = 2x^2 + x - 1$ and $s = -3$.

Polynomial root-finding

The **rational root theorem** tells us that for a polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ all of whose coefficients (a_n through a_0) are **integers**, the real rational roots are always of the form p/q, where p is an integer divisor of a_0 and q is an integer divisor of a_n. Thus if our polynomial is

$$P(x) = x^3 + 2x^2 - x - 2 = 0,$$

then the possible rational roots are all the integer divisors of a_0 (−2):

$$\text{Possible roots: } \{+1, -1, +2, -2\}\,.$$

So, setting *r* equal to each of these possible roots in turn, we will test-divide the polynomial by (*x* – *r*). If the resulting quotient has no remainder, we have found a root.

You can choose one of the following three methods: they will all yield the same results, with the exception that only through the second method and the third method (when applying Ruffini's rule to obtain a factorization) can you discover that a given root is repeated. (Remember that neither method will discover irrational or complex roots.)

Method 1

We try to divide *P*(*x*) by the binomial (*x* – each possible root). If the remainder is 0, the selected number is a root (and vice versa)

```
    |    +1     +2     -1      -2
    |
 +1 |           +1     +3      +2
----|---------------------------
    |    +1     +3     +2       0
```

```
    |    +1     +2     -1     -2
    |
 -1 |           -1     -1     +2
----|--------------------------
    |    +1     +1     -2      0
```

```
    |    +1     +2     -1      -2
    |
 +2 |           +2     +8     +14
----|---------------------------
    |    +1     +4     +7     +12
```

```
    |    +1     +2     -1     -2
    |
 -2 |           -2      0     +2
----|--------------------------
    |    +1      0     -1      0
```

$$x_1 = +1$$
$$x_2 = -1$$
$$x_3 = -2$$

Method 2

We start just as in Method 1 until we find a valid root. Then, instead of restarting the process with the other possible roots, we continue testing the possible roots against the result of the Ruffini on the valid root we've just found until we only have a coefficient remaining (remember that roots can be repeated: if you get stuck, try each valid root twice):

```
    |    +1     +2     -1     -2
    |
 -1 |           -1     -1     +2
----|----------------------------
    |    +1     +1     -2    | 0
    |
 +2 |           +2     +6
-------------------------
    |    +1     +3    |+4
```

```
    |    +1     +2     -1     -2
    |
 -1 |           -1     -1     +2
----|----------------------------
    |    +1     +1     -2    | 0
    |
 +1 |           +1     +2
-------------------------
    |    +1     +2    | 0
    |
 -2 |           -2
-------------------
    |    +1    | 0
```

$$x_1 = -1$$
$$x_2 = +1$$
$$x_3 = -2$$

Method 3

- Determine the set of the possible integer or rational roots of the polynomial according to the rational root theorem.
- For each possible root r, instead of performing the division P(x)/(x -r), apply the polynomial remainder theorem, which states that the remainder of this division is P(r), i.e. the polynomial evaluated for x = r.

Thus, for each r in our set, r is actually a root of the polynomial if and only if P(r) = 0

This shows that finding *integer and rational* roots of a polynomial neither requires any division nor the application of Ruffini's rule.

However, once a valid root has been found, call it r_1: you can apply Ruffini's rule to determine
$Q(x) = P(x)/(x-r_1)$.
This allows you to partially factorize the polynomial as
$P(x) = (x - r_1)\cdot Q(X)$

Any additional (rational) root of the polynomial is also a root of Q(x) and, of course, is still to be found among the possible roots determined earlier which have not yet been checked

Finding roots without applying Ruffini's Rule

$P(x) = x^3 + 2x^2 - x - 2$

Possible roots = {1, -1, 2, -2}

- $P(1) = 0 \rightarrow x_1 = 1$

- $P(-1) = 0 \rightarrow x_2 = -1$
- $P(2) = 12 \rightarrow 2$ is not a root of the polynomial

and the remainder of $(x^3 + 2x^2 - x - 2)/(x-2)$ is 12

- $P(-2) = 0 \rightarrow x_3 = -2$

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2) Remainder Theorem

If $P(x)$ is divided by $(x-r)$, the remainder is $P(r)$.

Ex. Find the remainder when $P(x)= 2x^3 - 3x^2 + x + 5$ is divided by $(x - 2)$

$P(2)=16-12+2+5 = 11$; Remainder=11

3) Factor Theorem

If $P(r)=0$ then $(x-r)$ is a factor of $P(x)$.

Ex. Find the remainder when $P(x)= 3x^3 - 2x^2 + 4x - 75$ is divided by $(x - 3)$ and evaluate if it is a factor of $P(x)$.

$P(3)= 81 - 18 + 12 - 75 = 0$; Remainder=0; $(x-3)$ is a factor of $P(x)$.

We can check it using Ruffini's rule:

$$
\begin{array}{c|cccc}
 & 3 & -2 & 4 & 75 \\
3 & & 9 & 21 & 75 \\
\hline
 & 3 & 7 & 25 & 0
\end{array}
$$

So we can write:

$(3x^3 - 2x^2 + 4x - 75) = (3x^2 - 7x + 25) \cdot (x - 3)$